COMPLETE REDUCIBILITY THEOREMS FOR MODULES OVER POINTED HOPF ALGEBRAS

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Dedicated to Susan Montgomery for her many contributions to Hopf algebras, ring theory, and service to the mathematical community

Abstract. We investigate the representation theory of a large class of pointed Hopf algebras, extending results of Lusztig and others. We classify all simple modules in a suitable category and determine the weight multiplicities; we establish a complete reducibility theorem in this category.

Introduction

The main achievements of the finite-dimensional representation theory of finite-dimensional complex semisimple Lie algebras include

(a) the parametrization of the simple finite-dimensional representations by their highest weights,
(b) the complete reducibility of any finite-dimensional representation,
(c) the determination of the weight multiplicities (and consequently, of the dimension) of a finite-dimensional highest weight representation.

These classical results of E. Cartan – part (a) 1913, and of H. Weyl – parts (b), (c) 1926, were generalized in many directions. They hold for Kac-Moody algebras, with symmetrizable generalized Cartan matrices, in the context of *integrable* modules from the category $\mathcal{O}$ instead of *finite-dimensional* ones, as shown by V. Kac in 1974; see [K].

Drinfeld and Jimbo introduced two quantum versions of the universal enveloping algebras of a finite-dimensional simple Lie algebra $\mathfrak{g}$, the formal deformation $U_\hbar(\mathfrak{g})$ [Dr1] and the q-analogue $U_q(\mathfrak{g})$ [Ji]. The representation theory of the q-analogue $U_q(\mathfrak{g})$ was worked out in [L1], where highest weight modules were introduced and parts (a) and (c) above were established. Drinfe'ld observed in [Dr3] that, since $U_\hbar(\mathfrak{g})$ and $U(\mathfrak{g})[[\hbar]]$ are isomorphic algebras by the Whitehead Lemma, their representation theories

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are equivalent. He also introduced a quantum Casimir operator to deal with complete reduciblity. Later, the representation theory of the q-analogue $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra, was developed in [L2], where analogues of the highest weight modules from [K] were introduced and a complete reducibility theorem was proved [L2, 6.2.2] using a quantum version of the Casimir operator.

Multiparameter quantized enveloping algebras also have been defined as formal deformations and as q-analogues, see [Re] and [OY] respectively. In the context of formal deformations, this is just a twist in the sense of Drinfeld of $U_\hbar(\mathfrak{g})$, hence their representation theories are equivalent; see also [LS]. Besides [OY], multiparameter q-analogues of enveloping algebras were considered in many papers, where the number of parameters and the group of group-likes is varied; consult [AE]. In the last few years, there was a revival of this question and the representation theories of 2-parameter deformations were studied; see [BW1, BW2, BGH1, BGH2], and [HPR] for more general cases and references therein.

In [AS3] a family of pointed Hopf algebras was introduced having a Cartan matrix of finite type as one of the inputs. This family contains the q-analogues $U_q(\mathfrak{g})$ and their multiparametric variants; in fact, they are close to them but one parameter of deformation for each connected component and more general linking relations are allowed. Indeed, the family contains also the parabolic Hopf subalgebras of the $U_q(\mathfrak{g})$'s and eventually, any pointed Hopf algebra with generic braiding and finitely generated abelian group of group-likes which is a domain of finite Gelfand-Kirillov dimension belongs to it [AA, AS3]. The goal of the present paper is to study the representation theory of these Hopf algebras and of their natural generalizations with arbitrary symmetrizable Cartan matrices. Let us summarize our main results.

In *Sections 1 and 2* we first consider very general pointed Hopf algebras $\mathcal{U}(\mathcal{D}, \lambda)$ (see Definition 1.9) defined for a general YD-datum $\mathcal{D}$ over an arbitray abelian group Γ and a linking parameter λ. A YD-datum is a realization of a diagonal braiding, without further restrictions. We prove a general structure result, Theorem 2.1, analogous to the classical Levi decomposition for Lie algebras.

We focus in *Section 3* on a special class of linking parameters, the perfect linkings. Following [RS2], we introduce the notions of reduced data $\mathcal{D}_{red}$ and their linking parameters ℓ. The Hopf algebras $\mathcal{U}(\mathcal{D}, \lambda)$ with perfect linking admit an alternative presentation in terms of reduced data: $\mathcal{U}(\mathcal{D}, \lambda) \simeq \mathcal{U}(\mathcal{D}_{red}, \ell)$; this stresses the similarity with $U_q(\mathfrak{g})$. Indeed, $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, \ell)$ (see Definition 3.3) has generators similar to those of $U_q(\mathfrak{g})$ (except for the group Γ of group-likes that is more general) and by Theorem 3.7 it is a quotient of a Drinfeld double. From this description we derive a basic bilinear form $(\ ,\ ) : \mathbf{U}^- \times \mathbf{U}^+ \to \Bbbk$. Once the existence of the form is shown, our approach to establish its main properties is close to previous work in the

literature, specially [L2]. We end this section with a discussion of data of Cartan type [AS3].

In *Section 4*, we study the representation theory of $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, \ell)$ with $\mathcal{D}_{red}$ generic (see Definition 1.4), regular (see Definition 3.18), and of Cartan type (see Definition 3.17).

The Hopf algebra $\mathbf{U} = \mathbf{U}_L$ in Lusztig's book [L2], where the root datum is X-regular, is a special case of our $\mathbf{U}$ above. The 2-parameter deformations mentioned above and the (generic) multiparameter deformations we have seen in the literature are all special cases of the Hopf algebra $\mathbf{U}$ in this Section.

We extend the results of Chapters 3, 4 and 6 in [L2] to our more general context following the strategy of [L2]. Similarly to [L2], we consider the category $\mathcal{C}$ of $\mathbf{U}$-modules with weight decomposition (with respect to the action of Γ), the full subcategory $\mathcal{C}^{hi}$ (see Subsection 4.1), and the class of integrable modules in $\mathcal{C}$ (see Subsection 4.2). However, Lusztig only considers representations of $\mathbf{U}_L$ with weights of the form χ_λ, where $\lambda \in X$, and $\chi_\lambda(K_\mu) = v^{<\mu,\lambda>}$ for all $\mu \in Y$, where the free abelian groups X, Y of finite rank are part of the given root datum in [L2]. The weights χ_λ ("of type one") do not make sense in our general context. Thus our category $\mathcal{C}$ for $\mathbf{U}_L$, where arbitrary elements in $\widehat{\Gamma}$ are allowed as weights, is larger than Lusztig's category $\mathcal{C}$ defined in [L2, 3.4.1].

Let $\widehat{\Gamma}^+$ be the set of all dominant characters $\chi \in \widehat{\Gamma}$ for $\mathcal{D}_{red}$ introduced in [RS2]. We define Verma modules $M(\chi)$ for all $\chi \in \widehat{\Gamma}$ and Weyl modules $L(\chi)$ for all $\chi \in \widehat{\Gamma}^+$. To define a version of the quantum Casimir operator of [L2, 6.11] we have to define a suitable scalar valued function on $\widehat{\Gamma}$ in Lemma 4.10 extending the integer valued function on X in [L2, Lemma 6.1.5] in the special case considered by Lusztig. It turns out that this is not always possible. But such a function exists if the Dynkin diagram of the generalized Cartan matrix of $\mathcal{D}_{red}$ is connected. We then reduce the later results to the connected case.

We call an algebra A reductive if all finite-dimensional left A-modules are semisimple. If $B \subset A$ is a subalgebra, we say that A is B-reductive if all finite-dimensional left A-modules which are semisimple over B are semisimple. We prove:

(α) By Corollary 4.16

$$\widehat{\Gamma}^+ \to \{[L] \mid L \in \mathcal{C}^{hi}, L \text{ integrable and simple}\},\ \chi \mapsto [L(\chi)],$$

is bijective. Here, $[L]$ denotes the isomorphism class of a module L. This establishes (a) in the context of integrable modules in $\mathcal{C}^{hi}$.

(β) By Theorem 4.17 any integrable module in $\mathcal{C}^{hi}$ is completely reducible. That is, we prove (b) in this context. This is one of our main results extending Lusztig's Theorem [L2, 6.2.2]. In particular, $\mathbf{U}$ is $\Bbbk\Gamma$-reductive.

(γ) Assume that the braiding is twist-equivalent to a braiding of Drinfeld-Jimbo type [AS3] (see 3.40). In particular this holds in the case of finite Cartan type. Then by Theorem 4.15 the weight multiplicities, in particular the dimension, of $L(\chi)$ with χ dominant, are as in the classical case. Thus (c) is shown.

(δ) By Theorem 4.21 $\mathbf{U}$ is reductive iff the index $[\Gamma : \Gamma^2]$ is finite, where Γ^2 is a subgroup of Γ given by the datum $\mathcal{D}_{red}$. Hence we have determined all Hopf algebras $\mathbf{U}$ satisfying (a) and (b) in the context of finite-dimensional modules.

In the case of 2-parameter deformations of finite Cartan type (α) and (β) have been shown in [BW2] for type A and in [BGH2] for types B, C and D; see also [HPR] for more general cases but under very restrictive assumptions on the braiding.

In *Section 5* we extend Theorem 4.21 to the pointed Hopf algebras $\mathcal{U}(\mathcal{D}, \lambda)$ in the case of finite Cartan type. We show in Theorem 5.3 that the linking is perfect if and only if $\mathcal{U}(\mathcal{D}, \lambda)$ is $\Bbbk\Gamma$-reductive. Our proof of this close relationship between reductivity and properties of the linking is based on the Levi type decomposition in Theorem 2.1 and recent results on PBW-bases in left coideal subalgebras of quantum groups in [Kh] or [HS]. Combined with the main results of [AS3, AA], our theory gives in Theorem 5.4 a characterization of the pointed Hopf algebras $\mathbf{U}$ with finite Cartan matrix and free abelian group of finite rank Γ by axiomatic properties.

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1. Nichols algebras and linking

We denote the ground field by $\Bbbk$, and its multiplicative group of units by $\Bbbk^\times$. We assume that $\Bbbk$ is algebraically closed of characteristic zero. By convention, $\mathbb{N} = \{0, 1, \dots\}$. If A is an algebra and $g \in A$ is invertible, then $g \triangleright a = gag^{-1}$, $a \in A$, denotes the inner automorphism defined by g.

We use standard notation for Hopf algebras; the comultiplication is denoted Δ and the antipode $\mathcal{S}$. For the first, we use the Heyneman-Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$. The left adjoint representation of H on itself is the algebra map $\operatorname{ad} : H \to \operatorname{End}(H)$, $\operatorname{ad}_l x(y) = x_{(1)} y \mathcal{S}(x_{(2)})$, $x, y \in H$; we shall write ad for ad_l, omitting the subscript l unless strictly needed. There is also a right adjoint action given by $\operatorname{ad}_r x(y) = \mathcal{S}(x_{(1)}) y x_{(2)}$. Note that both ad_l and ad_r are multiplicative.

1.1. **Yetter-Drinfeld modules and Nichols algebras.**

For a full account of these structures, the reader is referred to [AS1]. Let H be a Hopf algebra with bijective antipode. A Yetter-Drinfeld module over H is a left H-module and a left H-comodule with comodule structure denoted by $\delta : V \to H \otimes V, v \mapsto v_{(-1)} \otimes v_{(0)}$, such that

$$\delta(hv) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} v_{(0)}$$

for all $v \in V, h \in H$. Let ${}^H_H\mathcal{YD}$ be the category of Yetter-Drinfeld modules over H with H-linear and H-colinear maps as morphisms.

The category ${}^H_H\mathcal{YD}$ is monoidal and braided. If $V, W \in {}^H_H\mathcal{YD}$, then $V \otimes W$ is the tensor product over $\Bbbk$ with the diagonal action and coaction of H and braiding

$$c_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}$$

for all $v \in V, w \in W$. This allows us to consider Hopf algebras in ${}^H_H\mathcal{YD}$. If R is a Hopf algebra in the braided category ${}^H_H\mathcal{YD}$, then the space of primitive elements $P(R) = \{x \in R \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$ is a Yetter-Drinfeld submodule of R.

For $V \in {}^H_H\mathcal{YD}$ the tensor algebra $T(V) = \oplus_{n \geq 0} T^n(V)$ is an $\mathbb{N}$-graded algebra and coalgebra in the braided category ${}^H_H\mathcal{YD}$ where the elements of $V = T(V)(1)$ are primitive. It is a Hopf algebra in ${}^H_H\mathcal{YD}$ since $T(V)(0) = \Bbbk$.

We now recall the definition of the fundamental example of a Hopf algebra in a category of Yetter-Drinfeld modules.

Definition 1.1. Let $V \in {}^H_H\mathcal{YD}$ and $I(V) \subset T(V)$ the largest $\mathbb{N}$-graded ideal and coideal $I \subset T(V)$ such that $I \cap V = 0$. We call $\mathcal{B}(V) = T(V)/I(V)$ the *Nichols algebra* of V. Then $\mathcal{B}(V) = \oplus_{n \geq 0} \mathcal{B}^n(V)$ is an $\mathbb{N}$-graded Hopf algebra in ${}^H_H\mathcal{YD}$.

Lemma 1.2. *The Nichols algebra of an object $V \in {}^H_H\mathcal{YD}$ is (up to isomorphism) the unique $\mathbb{N}$-graded Hopf algebra R in ${}^H_H\mathcal{YD}$ satisfying the following properties:*

(1) $R(0) = \Bbbk,\ R(1) = V$,
(2) $R(1)$ *generates the algebra R,*
(3) $P(R) = V$.

If $f : V \to W$ is a morphism in ${}^H_H\mathcal{YD}$, then $T(f)(I(V)) \subset I(W)$, where $T(f) : T(V) \to T(W)$ is the induced map on the tensor algebras; hence f induces a morphism between the corresponding Nichols algebras.

Proof. [AS1, Proposition 2.2, Corollary 2.3]. □

Let Γ be an abelian group. A Yetter-Drinfeld module over the group algebra $\Bbbk\Gamma$ is a Γ-graded vector space $V = \oplus_{g \in \Gamma} V_g$ which is a left Γ-module such that each homogeneous component V_g, $g \in \Gamma$, is stable under the action of Γ. The Γ-grading is equivalently described as a left $\Bbbk\Gamma$-comodule structure $\delta : V \to \Bbbk\Gamma \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ where $\delta(v) = g \otimes v$ if v is homogeneous of degree $g \in \Gamma$. Let ${}^\Gamma_\Gamma\mathcal{YD}$ be the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$. For $V, W \in {}^\Gamma_\Gamma\mathcal{YD}$ the braiding is given by $c_{V,W}(v \otimes w) = g \cdot w \otimes v$ for all $v \in V_g$, $g \in \Gamma$, $w \in W$.

1.2. **Braided Hopf algebras and bosonization.**

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. We denote its comultiplication by $\Delta_R : R \to R \otimes R,\ r \mapsto r^{(1)} \otimes r^{(2)}$. The *bosonization* $R\#H$ of R is a

Hopf algebra defined as follows. As a vector space, $R\#H = R\otimes H$; the multiplication and comultiplication of $R\#H$ are given by the smash-product and smash-coproduct, respectively, that is, for all $r,s\in R, g,h\in H$,

$$(r\otimes g)(s\otimes h) = r(g_{(1)}\cdot s)\otimes g_{(2)}h,$$
$$\delta(r\otimes g) = r^{(1)}\#(r^{(2)})_{(-1)}g_{(1)}\otimes (r^{(2)})_{(0)}\#g_{(2)}.$$

Let $\pi : A\to H$ be a morphism of Hopf algebras. Then

$$R = A^{\operatorname{co} H} = \{a\in A \mid (\operatorname{id}\otimes\pi)\Delta(a) = a\otimes 1\}$$

is the left coideal subalgebra of right coinvariants of π. There is also a left version: ${}^{\operatorname{co} H}A = \{a\in A : (\pi\otimes\operatorname{id})\Delta(a) = 1\otimes a\}$. The subalgebra $A^{\operatorname{co} H}\subset A$ is stable under the left adjoint action ad_l of A, and ${}^{\operatorname{co} H}A$ is stable under ad_r.

Lemma 1.3. *Let $\pi : A\to H$ be a morphism of Hopf algebras, and assume that the antipode $\mathcal{S}$ of A is bijective. Then*

(1) $\mathcal{S}(A^{\operatorname{co} H}) = {}^{\operatorname{co} H}A$.

(2) *Assume that there is a Hopf algebra map $\iota : H\to A$ such that $\pi\iota = \operatorname{id}_H$. Then the map*

$$\kappa : A^{\operatorname{co} H}\to {}^{\operatorname{co} H}A,\ a\mapsto \mathcal{S}^2(a_{(2)})\mathcal{S}(\iota\pi(a_{(1)})),$$

is bijective, and for all $x,y\in A$, $\kappa(xy) = \kappa(x_{(2)})\operatorname{ad}_r\iota\pi\mathcal{S}(x_{(1)})(\kappa(y))$.

Proof. This is easily checked. The map $b\mapsto \mathcal{S}^{-2}(b_{(1)})\mathcal{S}^{-1}(\iota\pi(b_{(2)}))$ is inverse to κ. □

Assume the situation of part (2) of Lemma 1.3. Then $R = A^{\operatorname{co} H}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, and the multiplication induces an isomorphism $R\#H\to A, r\#h\mapsto r\iota(h)$, of Hopf algebras.

Conversely any Hopf algebra R in ${}^H_H\mathcal{YD}$ arises in this way from the bosonization since $\pi = \varepsilon\otimes\operatorname{id} : R\#H\to H$ is a Hopf algebra map with $(R\#H)^{\operatorname{co} H} = R\otimes 1$. The braided adjoint action $\operatorname{ad}_c : R\to\operatorname{End}(R)$ is defined for all $x,y\in R$ by $\operatorname{ad}_c x(y) = \operatorname{ad} x(y)$, where ad is the left adjoint action of the bosonization $R\#H$, and where we view $R\to R\#H$, $r\mapsto r\#1$, as inclusion. In particular, if $x\in P(R)$, then $\operatorname{ad}_c x(y) = xy - (x_{(-1)}\cdot y)x_{(0)}$ is the braided commutator of x and y.

1.3. **Yetter-Drinfeld data.**

We are interested in finite-dimensional Yetter-Drinfeld modules over an abelian group Γ that are semisimple as Γ-modules; these are described in the following way.

Definition 1.4. A *YD-datum* $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$ consists of an abelian group Γ, a positive integer θ, $g_1,\dots,g_\theta\in\Gamma$, and characters $\chi_1,\dots,\chi_\theta\in\widehat{\Gamma} = \operatorname{Hom}(\Gamma,\Bbbk^\times)$. We let $\mathbb{I} = \{1,2,\dots,\theta\}$ and define

$$q_{ij} = \chi_j(g_i) \text{ for all } i,j\in\mathbb{I}. \tag{1.1}$$

A YD-datum is called *generic* if for all $1 \le i \le \theta$, q_{ii} is not a root of unity. We define an equivalence relation $\sim$ on $\mathbb{I}$, where for all $i, j \in \mathbb{I}$, $i \neq j$,

$$i \sim j \iff \begin{array}{l} \text{There are } i_1, \dots, i_t \in \mathbb{I}, t \ge 2 \text{ with } i = i_1, j = i_t, \\ q_{i_l i_{l+1}} q_{i_{l+1} i_l} \neq 1 \text{ for all } 1 \le l < t. \end{array} \tag{1.2}$$

Let $\mathcal{X}$ be the set of equivalence classes of $\mathbb{I}$ with respect to $\sim$.

Remark 1.5. The notions of YD-datum and the equivalence relation (1.2) are generalizations of the notions of Cartan datum and the resulting equivalence relation from [RS2, Definition 3.16].

Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \le i \le \theta}, (\chi_i)_{1 \le i \le \theta})$ be a YD-datum. Let X be a vector space with basis $x_1, \dots, x_\theta$. Then $\mathcal{D}$ defines on X a Yetter-Drinfeld module structure over $\Bbbk\Gamma$ where for all $i \in \mathbb{I}$, $g \in \Gamma$,

$$\delta(x_i) = g_i \otimes x_i, \tag{1.3}$$

$$g x_i = \chi_i(g) x_i. \tag{1.4}$$

Then the braiding $c = c_{X,X}$ of X is given by

$$c : X \otimes X \to X \otimes X, \quad c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, 1 \le i, j \le \theta.$$

We identify the tensor algebra $T(X)$ with the free algebra $\Bbbk\langle x_1, \dots, x_\theta \rangle$.

Let $\mathbb{Z}^{\mathbb{I}}$ be a free abelian group of rank θ with fixed basis $\alpha_1, \dots, \alpha_\theta$, and $\mathbb{N}^{\mathbb{I}} = \{\alpha = \sum_{i=1}^{\theta} n_i \alpha_i \mid n_1, \dots, n_\theta \in \mathbb{N}\}$. The homogeneous components of a $\mathbb{Z}^{\mathbb{I}}$-graded vector space Z will be denoted by Z_α, $\alpha \in \mathbb{Z}^{\mathbb{I}}$. The tensor algebra $T(X)$ is an $\mathbb{N}^{\mathbb{I}}$-graded algebra where each x_i has degree α_i.

Lemma 1.6.
(1) *$I(X)$ is an $\mathbb{N}^{\mathbb{I}}$-graded ideal of $T(X)$, and $\mathcal{B}(X)$ is an $\mathbb{N}^{\mathbb{I}}$-graded algebra and coalgebra.*
(2) *Let $i, j \in \mathbb{I}$, $i \neq j$, and assume $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for some integer a_{ij} with $0 \le -a_{ij} < \operatorname{ord}(q_{ii})$ (where $1 \le \operatorname{ord}(q_{ii}) \le \infty$). Then*

$$\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0 \text{ in } \mathcal{B}(X). \tag{1.5}$$

Proof. (2) is shown in [AS1, 3.7]. For (1) see for example [AHS, Remark 2.8]. □

We extend the notion of linking parameter given in [AS3] for data of finite Cartan type, to the general case treated here.

Definition 1.7. Vertices $i, j \in \mathbb{I}$ are called *linkable* if

$$i \not\sim j, \tag{1.6}$$

$$g_i g_j \neq 1, \tag{1.7}$$

$$\chi_i \chi_j = 1 \text{ (the trivial character)}. \tag{1.8}$$

A family $\lambda = (\lambda_{ij})_{i,j\in\mathbb{I}, i\not\sim j}$ of elements in $\Bbbk$ is called a *linking parameter* for $\mathcal{D}$ if for all $i, j \in \mathbb{I}$, $i \not\sim j$,

$$\lambda_{ij} = 0 \quad \text{if } i, j \text{ are not linkable,} \tag{1.9}$$

$$\lambda_{ij} = -q_{ij}\lambda_{ji}. \tag{1.10}$$

Given a linking parameter λ for $\mathcal{D}$, vertices $i, j \in \mathbb{I}, i \not\sim j$, are called *linked* if $\lambda_{ij} \neq 0$.

The next Lemma generalizes [AS2, Lemma 5.6]. We include the proof for completeness.

Lemma 1.8. (1) *Let $i, j, k, l \in \mathbb{I}$.*

(a) *If i, k are linkable, then $q_{ji}q_{jk} = 1$, and $q_{ii} = q_{kk}^{-1} = q_{ki} = q_{ik}^{-1}$.*

(b) *If i, k and j, l are linkable, then $q_{ij}q_{ji}q_{kl}q_{lk} = 1$.*

(2) *Assume that*

$$q_{ij}q_{ji} \neq q_{ii}^2 \text{ for all } i, j \in \mathbb{I}, i \neq j. \tag{1.11}$$

Then any vertex $i \in \mathbb{I}$ is linkable to at most one $k \in \mathbb{I}$.

Proof. (1) (a) follows easily from (1.8) since $q_{ik}q_{ki} = 1$ by (1.6). (b) Since i, k and j, l are linkable, $q_{ij} = q_{il}^{-1}, q_{ji} = q_{jk}^{-1}, q_{kl} = q_{kj}^{-1}, q_{lk} = q_{li}^{-1}$ by (1.8). Hence

$$q_{ij}q_{ji}q_{kl}q_{lk} = (q_{il}q_{li})^{-1}(q_{jk}q_{kj})^{-1}. \tag{1.12}$$

If $i \sim l$ or $j \sim k$, then $i \not\sim j$ and $k \not\sim l$ since by assumption $i \not\sim k$ and $j \not\sim l$. Thus the LHS of (1.12) is equal to 1 by (1.2). And if $i \not\sim l$ and $j \not\sim k$, then the RHS of (1.12) is equal to 1 by (1.2). This proves the claim.
(2) If $i \in \mathbb{I}$ is linkable to $k \in \mathbb{I}$ and to $l \in \mathbb{I}$, then $q_{ii}^2 q_{kl}q_{lk} = 1$ by (1)(b), and $q_{ii} = q_{kk}^{-1}$ by (1)(a). Hence $q_{kl}q_{lk} = q_{kk}^2$, and $k = l$ by assumption. $\square$

For any subset $J \subset \mathbb{I}$, let $X_J = \oplus_{j\in J}\Bbbk x_j \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. Recall the ideal $I(V)$ in Definition 1.1.

Let λ be a linking parameter for the YD-datum $\mathcal{D}$.

Definition 1.9.

$$\mathcal{U}(\mathcal{D}, \lambda) = (T(X)\#\, \Bbbk\Gamma)/I, \tag{1.13}$$

where $I \subset T(X)\#\Bbbk\Gamma$ is the ideal generated by

$$I(X_J) \text{ for all } J \in \mathcal{X}, \tag{1.14}$$

$$x_ix_j - q_{ij}x_jx_i - \lambda_{ij}(1 - g_ig_j) \text{ for all } i, j \in \mathbb{I}, i \not\sim j. \tag{1.15}$$

Then $\mathcal{U}(\mathcal{D}, \lambda)$ is a Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with comultiplication given by

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1,\ 1 \leq i \leq \theta, \tag{1.16}$$

$$\Delta(g) = g \otimes g,\ g \in \Gamma. \tag{1.17}$$

By abuse of language we identify $x \in X$ and $g \in \Gamma$ with their images in $\mathcal{U}(\mathcal{D}, \lambda)$.

Remark 1.10. The ideal $I(X) \subset T(X)$ is generated by

(1.18) $\quad I(X_J)$ for all $J \in \mathcal{X}$,

(1.19) $\quad x_i x_j - q_{ij} x_j x_i$ for all $i, j \in \mathbb{I}, i \not\sim j$.

Hence $\mathcal{U}(\mathcal{D}, 0) \cong \mathcal{B}(X) \# \Bbbk\Gamma$.

Generalizing part of [AS3, Theorem 4.3], Masuoka proved the following result.

Theorem 1.11. [Ma, 5.2] *Let $\mathcal{X} = \{J_1, \dots, J_t\}$, $t \geq 1$, $J_i \neq J_j$ for $i \neq j \in \mathbb{I}$. For $1 \leq l \leq t$ let $X_l = X_{J_l}$, and let $\rho_l : \mathcal{B}(X_l) \to \mathcal{U}(\mathcal{D}, \lambda)$ be the canonical map induced by the inclusion $T(X_l) \subset T(X)$. Then the linear map*

$$\mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_t) \otimes \Bbbk\Gamma \to \mathcal{U}(\mathcal{D}, \lambda),$$
$$r_1 \otimes \dots r_t \otimes g \mapsto \rho_1(r_1) \cdots \rho_t(r_t) g$$

is bijective. □

Masuoka even showed that the isomorphism in Theorem 1.11 is a coalgebra isomorphism inducing a Hopf algebra structure on

$$(\mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_t)) \# \Bbbk\Gamma$$

which is a 2-cocycle deformation. In fact he only assumes that the $\mathcal{B}(X_l)$ are pre-Nichols algebras – satisfying (1), (2), in Lemma 1.2 – and uses a more general equivalence relation.

We close this section with a technical lemma that will be used later.

Lemma 1.12. *Let $j, i_1, \dots, i_n \in \mathbb{I}, n \geq 1, j \not\sim i_1, \dots, j \not\sim i_n$. Then in $\mathcal{U}(\mathcal{D}, \lambda)$,*

$$x_j x_{i_1} \cdots x_{i_n} = q_{i_1 j} \cdots q_{i_n j} x_{i_1} \cdots x_{i_n} x_j + \sum_{1 \leq \nu \leq n} q_{i_1 j} \cdots q_{i_{\nu-1} j} x_{i_1} \cdots x_{i_{\nu-1}} \lambda_{j i_\nu} (1 - g_j g_{i_\nu}) x_{i_{\nu+1}} \cdots x_{i_n}.$$

Proof. If $j \in \mathbb{I}, x \in \mathcal{U}(\mathcal{D}, \lambda)$, then $x_j x = \operatorname{ad} g_j(x) x_j + \operatorname{ad} x_j(x)$ by the definition of ad; indeed, $\Delta(x_j) = g_j \otimes x_j + x_j \otimes 1$, hence $\mathcal{S}(x_j) = -g_j^{-1} x_j$. We apply this formula with $x = x_{i_1} \cdots x_{i_n}$ and obtain

$$x_j x_{i_1} \cdots x_{i_n} = q_{i_1 j} \cdots q_{i_n j} x_{i_1} \cdots x_{i_n} + \operatorname{ad} x_j (x_{i_1} \dots x_{i_n}).$$

Now

$$\begin{aligned}
\operatorname{ad} x_j (x_{i_1} \cdots x_{i_n}) &= \operatorname{ad} x_{j(1)}(x_{i_1}) \cdots \operatorname{ad} x_{j(n)}(x_{i_n}) \\
&= \sum_{1 \leq \nu \leq n} \operatorname{ad} g_j(x_{i_1}) \cdots \operatorname{ad} g_j(x_{i_{\nu-1}}) \operatorname{ad} x_j(x_{i_\nu}) x_{i_{\nu+1}} \cdots x_{i_n} \\
&= \sum_{1 \leq \nu \leq n} q_{i_1 j} \cdots q_{i_{\nu-1} j} x_{i_1} \cdots x_{i_{\nu-1}} \lambda_{j i_\nu} (1 - g_j g_{i_\nu}) x_{i_{\nu+1}} \cdots x_{i_n},
\end{aligned}$$

where we used $j \not\sim i_\nu$ in the third equality, and the claim follows. □

2. A Levi-type theorem for pointed Hopf algebras

Let $\mathcal{D}$ be a YD-datum with linking parameter λ. We study the situation when unlinked vertices are omitted. Let

$\mathbb{I}^{\mathrm{s}} = \{h \in \mathbb{I} : h \text{ is not linked}\}$;
L a subset of $\mathbb{I}^{\mathrm{s}}$;
$\mathbb{I}' = \mathbb{I} - L$;
$X' = X_{\mathbb{I}'}$;
$\mathcal{D}' = \mathcal{D}(\Gamma, (g_i)_{i\in\mathbb{I}'}, (\chi_i)_{i\in\mathbb{I}'})$;
$\approx$, the equivalence relation on $\mathbb{I}'$ defined by the YD-datum $\mathcal{D}'$;
$\lambda'_{ij} = \begin{cases} \lambda_{ij}, & \text{if } i \not\sim j \\ 0, & \text{if } i \sim j, \end{cases}$ for all $i, j \in \mathbb{I}', i \not\approx j$.

Then λ' is a linking parameter for $\mathcal{D}'$, since λ is a linking parameter for $\mathcal{D}$. The inclusion $\iota : X' \to X$ has a section $\pi : X \to X'$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ defined by

$$x_i \overset{\pi}{\longmapsto} x_i, \quad , x_h \overset{\pi}{\longmapsto} 0, \quad i \in \mathbb{I}',\, h \in L.$$

Our next Theorem can be viewed as a "quantum version" of the classical Levi theorem for Lie algebras, see for instance [D, 1.6.9]. We shall investigate the case when $L = \mathbb{I}^{\mathrm{s}}$ in the next section.

Theorem 2.1. *The maps π and ι induce Hopf algebra morphisms $\Phi : \mathcal{U}(\mathcal{D}, \lambda) \to \mathcal{U}(\mathcal{D}', \lambda')$ and $\Psi : \mathcal{U}(\mathcal{D}', \lambda') \to \mathcal{U}(\mathcal{D}, \lambda)$ with $\Phi\Psi = \mathrm{id}$. Then $K = \mathcal{U}(\mathcal{D}, \lambda)^{\mathrm{co}\,\Phi}$ is a braided Hopf algebra in ${}^{\mathcal{U}(\mathcal{D}',\lambda')}_{\mathcal{U}(\mathcal{D}',\lambda')}\mathcal{YD}$ and there is an isomorphism*

$$K\#\mathcal{U}(\mathcal{D}', \lambda') \cong \mathcal{U}(\mathcal{D}, \lambda),$$

given by multiplication. Furthermore, the algebra K is generated by the set

$$S = \{\mathrm{ad}(x_{i_1} \cdots x_{i_n})(x_h) \mid h \in L,\, n \geq 0,\, i_\nu \in \mathbb{I}',\, i_\nu \sim h, 0 \leq \nu \leq n\}.$$

Proof. Let $\mathcal{U} = \mathcal{U}(\mathcal{D}, \lambda)$, $\mathcal{U}' = \mathcal{U}(\mathcal{D}', \lambda')$ for brevity.

Existence of Ψ. We have to show that the inclusion $\iota : T(X')\#\Bbbk\Gamma \hookrightarrow T(X)\#\Bbbk\Gamma$ maps the relations of $\mathcal{U}'$ to the relations of $\mathcal{U}$. Let J' be an equivalence class of $\approx$. Then there is exactly one equivalence class J of $\sim$ with $J' \subset J$. By Lemma 1.2, $\iota(I(X_{J'})) \subset I(X_J)$. Let $i, j \in \mathbb{I}', i \not\approx j$. We have to show that $\iota(x_i x_j - q_{ij} - \lambda'_{ij}(1 - g_i g_j))$ is a relation of $\mathcal{U}$. This is clear if $i \not\sim j$ since $\lambda'_{ij} = \lambda_{ij}$ in this case. If $i \sim j$ then $\lambda'_{ij} = 0$ by definition, and the relation $x_i x_j - q_{ij} x_j x_i = 0$ holds in $\mathcal{U}$ by (1.5) since $q_{ij} q_{ji} = 1$ follows from $i \not\approx j$.

Existence of Φ. Now we show that the projection $\pi : T(X)\#\Bbbk\Gamma \to T(X')\#\Bbbk\Gamma$ preserves the relations. Let J be an equivalence class of $\sim$ in $\mathbb{I}$, and $f \in I(X_J) \subset T(X)$; we may assume that f is $\mathbb{N}^{\mathbb{I}}$-homogeneous by Lemma 1.6 (1). If f does not contain any variable x_h, $h \in L$, then $\pi(f) = f$. Hence $f \in I(X')$ by Lemma 1.2. Thus $f \in T(X_J)$ is contained in the ideal $I(X')$ of $T(X')$; but this is generated by elements in $I(X_{J'})$, J' an equivalence class of $\approx$, and elements of the form $x_i x_j - q_{ij} x_j x_i$, where

$i, j \in \mathbb{I}', i \not\approx j$, and where we can assume that $i, j \in J$. Since $\lambda'_{ij} = 0$ for all $i, j \in \mathbb{I}', i \not\approx j, i \sim j$, it follows that f is in the ideal generated by the defining relations of $\mathcal{U}'$. If f does contain a variable x_h, where $h \in L$, then $\pi(f) = 0$ since $\pi(x_h) = 0$.

Finally, let $i, j \in \mathbb{I}, i \not\sim j$. If $i \in L$ or $j \in L$, then

$$\pi(x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(1 - g_i g_j)) = 0,$$

since $\lambda_{ij} = 0$ (because no vertex in L is linked). If $i \notin L, j \notin L$, then $i \not\approx j$, and $\lambda_{ij} = \lambda'_{ij}$. Hence $\pi(x_i x_j - q_{ij} x_j x_i - \lambda_{ij}(1 - g_i g_j)) = x_i x_j - q_{ij} x_j x_i - \lambda'_{ij}(1 - g_i g_j)$ is a relation of $\mathcal{U}'$.

Since $\Phi\Psi = \mathrm{id}$ (because this holds on the generators), the multiplication map $\mu : K \# \mathcal{U}' \to \mathcal{U}$ is an isomorphism. Let $\widetilde{K}$ be the subalgebra of K generated by S. Suppose we have shown that $\widetilde{K}\mathcal{U}'$ is a subalgebra of $\mathcal{U}$. Then $\widetilde{K}\mathcal{U}' = \mathcal{U}$ since $\widetilde{K}\mathcal{U}'$ contains the generators $g \in \Gamma$ and $x_i, i \in \mathbb{I}$ of the algebra $\mathcal{U}$. Since μ is bijective, $\widetilde{K} = K$.

To prove that $\widetilde{K}\mathcal{U}'$ is a subalgebra of $\mathcal{U}$, we have to show that

$$x_j \widetilde{K} \subset \widetilde{K}\mathcal{U}' \text{ for all } j \in \mathbb{I}'. \tag{2.1}$$

Then the claim follows easily by induction since the elements $x_{j_1} \dots x_{j_n} g$, $j_1, \dots, j_n \in \mathbb{I}', n \geq 0, g \in \Gamma$, generate $\mathcal{U}'$ as a vector space, and $g\widetilde{K} = \widetilde{K}g$. To prove (2.1) it is enough to show that

$$x_j \operatorname{ad}(x_{i_1} \cdots x_{i_n})(x_h) \in \widetilde{K}\mathcal{U}' \tag{2.2}$$

for all $j \in \mathbb{I}'$, $i_1, \dots, i_n \in \mathbb{I}$, $n \geq 0$, $h \in L$, $i_1 \sim h, \dots, i_n \sim h$. Let $x = \operatorname{ad}(x_{i_1} \cdots x_{i_n})(x_h)$. By the beginning of the proof of Lemma 1.12,

$$x_j x = q_{i_1 j} \cdots q_{i_n j} q_{hj} x x_j + \operatorname{ad} x_j(x),$$

and it remains to show that $\operatorname{ad} x_j(x) = \operatorname{ad}(x_j x_{i_1} \cdots x_{i_n})(x_h) \in \widetilde{K}\mathcal{U}'$. This is clear by definition of S if $j \sim h$. If $j \not\sim h$, then $\lambda_{jh} = 0$ since $h \in L$, and $\operatorname{ad} x_j(x_h) = 0$. By Lemma 1.12,

$$\begin{aligned}
\operatorname{ad}(x_j x_{i_1} \cdots x_{i_n})(x_h) &= \operatorname{ad}(q_{i_1 j} \cdots q_{i_n j} x_{i_1} \cdots x_{i_n}) \operatorname{ad} x_j(x_h) \\
&\quad + \operatorname{ad}\Big(\sum_{1 \leq \nu \leq n} q_{i_1 j} \cdots q_{i_{\nu-1} j} x_{i_1} \cdots x_{i_{\nu-1}} \lambda_{j i_\nu}(1 - g_j g_{i_\nu}) x_{i_{\nu+1}} \dots x_{i_n}\Big)(x_h) \\
&= \operatorname{ad}\Big(\sum_{1 \leq \nu \leq n} q_{i_1 j} \cdots q_{i_{\nu-1} j} x_{i_1} \cdots x_{i_{\nu-1}} \lambda_{j i_\nu}(1 - g_j g_{i_\nu}) x_{i_{\nu+1}} \dots x_{i_n}\Big)(x_h)
\end{aligned}$$

is a $\Bbbk$-linear combination of elements in S. □

Corollary 2.2. *In the situation of Theorem 2.1, let $\mathcal{U} = \mathcal{U}(\mathcal{D}, \lambda)$, $\mathcal{U}' = \mathcal{U}(\mathcal{D}', \lambda')$ and*

$$M = \mathcal{U}/(\mathcal{U}\mathcal{U}'^{+} + \mathcal{U}(K^{+})^2),$$

where $\mathcal{U}^{+}$ and K^{+} are the augmentation ideals with respect to the counit ε. Then $x_h M \neq 0$ for any $h \in L$.

Proof. By Theorem 2.1, the multiplication map $K \otimes \mathcal{U}' \to \mathcal{U}$ is bijective; let $\psi : \mathcal{U} \to K \otimes \mathcal{U}'$ be its inverse and

$$\varphi : \mathcal{U} \xrightarrow{\psi} K \otimes \mathcal{U}' \xrightarrow{\mathrm{id} \otimes \varepsilon} K.$$

Note that $\mathcal{U}K^+ = K^+\mathcal{U}$, since $K = \mathcal{U}^{\mathrm{co}\,\Phi}$ and the antipode $\mathcal{S}$ of $\mathcal{U}$ is bijective. For, K^+ is a submodule under ad_l and ad'_r, where $\mathrm{ad}'_r(u)(x) = \mathcal{S}^{-1}(u_{(2)})xu_{(1)}$, $x, u \in \mathcal{U}$; and the formulas

$$ux = (\mathrm{ad}_l(u_{(1)})(x))u_{(2)}, \qquad xu = u_{(2)}(\mathrm{ad}'_r(u_{(1)})(x)),$$

hold for $x, u \in \mathcal{U}$. Assume $x_h M = 0$. Then $x_h \in \mathcal{U}\mathcal{U}'^+ + \mathcal{U}(K^+)^2 = K\mathcal{U}'^+ + (K^+)^2\mathcal{U}'$. Since $x_h \in K$, it follows that

$$x_h = \varphi(x_h) \in (K^+)^2.$$

Thus by Theorem 2.1, x_h is the $\Bbbk$-span of products with at least two factors of the form $\mathrm{ad}\, x_{i_1} \cdots \mathrm{ad}\, x_{i_n}(x_h)$, $n \geq 0$, $i_1, \dots, i_n \in J$, where J is the connected component containing h. Since the Nichols algebra $\mathcal{B}(V_J)$ of $V_J = \oplus_{i \in J} \Bbbk x_i$ can be identified with the subalgebra of U generated by the elements $x_i, i \in J$ by Theorem 1.11, the element $x_h \in \mathcal{B}(V_J)$ has degree ≥ 2 which is impossible. □

3. Perfect linkings and reduced data

The goal of this section is to study a class of pointed Hopf algebras that resembles the quantized enveloping algebras $U_q(\mathfrak{g})$.

Definition 3.1. A linking parameter of a YD-datum $\mathcal{D}$ is *perfect* if and only if any vertex is linked.

By Theorem 2.1 for any linking parameter λ the Hopf algebra $\mathcal{U}(\mathcal{D}, \lambda)$ has a natural quotient Hopf algebra $\mathcal{U}(\mathcal{D}', \lambda')$ with perfect linking parameter λ'. This is the special case where $L = \mathbb{I}^s$ is the set of all non-linked vertices.

3.1. **Reduced data.** We begin with an alternative presentation of the Hopf algebra $\mathcal{U}(\mathcal{D}, \lambda)$ with perfect linking parameter; this stresses the similarity with quantized enveloping algebras.

Definition 3.2. A *reduced YD-datum*

$$\mathcal{D}_{red} = \mathcal{D}(\Gamma, (L_i)_{1\leq i\leq\theta}, (K_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta})$$

consists of an abelian group Γ, a positive integer θ, and elements $K_i, L_i \in \Gamma, \chi_i \in \widehat{\Gamma}$ for all $1 \leq i \leq \theta$ satisfying

$$\chi_j(K_i) = \chi_i(L_j) \text{ for all } 1 \leq i, j \leq \theta, \tag{3.1}$$

$$K_i L_i \neq 1 \text{ for all } 1 \leq i \leq \theta. \tag{3.2}$$

A reduced YD-datum $\mathcal{D}_{red}$ is called *generic* if for all $1 \leq i \leq \theta$, $\chi_i(K_i)$ is not a root of unity. A *linking parameter* ℓ for a reduced YD-datum $\mathcal{D}_{red}$ is a family $\ell = (\ell_i)_{1\leq i\leq\theta}$ of non-zero elements in $\Bbbk$.

Definition 3.3. Let $\mathcal{D}_{red} = \mathcal{D}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$ be a reduced YD-datum with linking parameter $\ell = (\ell_i)_{1\le i\le\theta}$. Let

$$V = \oplus_{i=1}^{\theta} \Bbbk v_i \in {}_{\Gamma}^{\Gamma}\mathcal{YD} \text{ with basis } v_i \in V_{K_i}^{\chi_i}, 1 \le i \le \theta, \tag{3.3}$$

$$W = \oplus_{i=1}^{\theta} \Bbbk w_i \in {}_{\Gamma}^{\Gamma}\mathcal{YD} \text{ with basis } w_i \in W_{L_i}^{\chi_i^{-1}}, 1 \le i \le \theta. \tag{3.4}$$

Then we define $\mathcal{U}(\mathcal{D}_{red}, \ell)$ as the quotient Hopf algebra of the biproduct $T(V \oplus W)\#\Bbbk\Gamma$ modulo the ideal generated by

$$I(V), \tag{3.5}$$

$$I(W), \tag{3.6}$$

$$v_i w_j - \chi_j^{-1}(K_i) w_j v_i - \delta_{ij}\ell_i(K_i L_i - 1) \text{ for all } 1 \le i, j \le \theta. \tag{3.7}$$

To a reduced YD-datum $\mathcal{D}_{red}$ with linking parameter ℓ we associate a YD-datum $\widetilde{\mathcal{D}_{red}}$ and a linking parameter $\widetilde{\ell}$ for $\widetilde{\mathcal{D}_{red}}$ by

$$\widetilde{\mathcal{D}_{red}} = \mathcal{D}(\Gamma, (\widetilde{g}_i)_{1\le i\le 2\theta}, (\widetilde{\chi}_i)_{1\le i\le 2\theta}), \text{ where} \tag{3.8}$$

$$(\widetilde{g}_1, \dots, \widetilde{g}_{2\theta}) = (L_1, \dots, L_\theta, K_1, \dots, K_\theta), \tag{3.9}$$

$$(\widetilde{\chi}_1, \dots, \widetilde{\chi}_{2\theta}) = (\chi_1^{-1}, \dots, \chi_\theta^{-1}, \chi_1, \dots, \chi_\theta), \tag{3.10}$$

$$\widetilde{\ell}_{i+\theta\, j} = -\delta_{ij}\ell_i \text{ for all } 1 \le i, j \le \theta, \tag{3.11}$$

$$\widetilde{\ell}_{kl} = 0 \text{ for all } 1 \le k, l \le 2\theta, k \not\approx l, k > l. \tag{3.12}$$

Here $\approx$ denotes the equivalence relation of $\widetilde{\mathcal{D}_{red}}$. Note that by (1.10) it suffices to define a linking parameter $(\widetilde{\ell}_{kl})$ for all $k > l$. Let $\widetilde{q}_{kl} = \widetilde{\chi}_l(\widetilde{g}_k)$ for all $1 \le k, l \le 2\theta$, and $q_{ij} = \chi_j(K_i)$ for all $1 \le i, j \le \theta$. Then it follows from (3.1) that for all $1 \le i, j \le \theta$,

$$\begin{aligned} \widetilde{q}_{ij}\widetilde{q}_{ji} &= (q_{ij}q_{ji})^{-1}, \\ \widetilde{q}_{\theta+i,\theta+j}\widetilde{q}_{\theta+j,\theta+i} &= q_{ij}q_{ji}, \\ \widetilde{q}_{i,\theta+j}\widetilde{q}_{\theta+j,i} &= 1. \end{aligned}$$

In particular, $i \not\approx \theta + j$ for all $1 \le i, j \le \theta$. Since $K_i L_i \ne 1$ by (3.2), it follows that $\widetilde{\ell}$ is a linking parameter for $\widetilde{\mathcal{D}_{red}}$.

Lemma 3.4. *Let $\mathcal{D}_{red} = \mathcal{D}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$ be a reduced YD-datum with linking parameter ℓ. Then*

$$\mathcal{U}(\mathcal{D}_{red}, \ell) \cong \mathcal{U}(\widetilde{\mathcal{D}_{red}}, \widetilde{\ell}).$$

Proof. This follows from the defining relations using Remark 1.10. □

Lemma 3.5. *Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$ be a YD-datum satisfying* (1.11)*, and let λ be a perfect linking parameter for $\mathcal{D}$. Then there is a reduced YD-datum $\mathcal{D}_{red}$ and a linking parameter ℓ for $\mathcal{D}_{red}$ such that*

$$\mathcal{U}(\mathcal{D}, \lambda) \cong \mathcal{U}(\mathcal{D}_{red}, \ell)$$

as Hopf algebras.

Proof. If $i \in \mathbb{I}$ then by Lemma 1.8 (2) there exists a unique $i^0 \in \mathbb{I}$ such that i and i^0 are linked. Thus $\mathbb{I} \to \mathbb{I},\ i \mapsto i^0$, is an involution on the set of vertices. By Lemma 1.8 (1)(b), $q_{ij}q_{ji}q_{i^0j^0}q_{j^0i^0} = 1$ for all $i, j \in \mathbb{I}$. Hence

$$\mathcal{X} \to \mathcal{X}, \quad J \mapsto J^0 = \{j^0 \mid j \in J\},$$

is an involution on the set of equivalence classes, and $J \cap J^0 = \emptyset$ for all $J \in \mathcal{X}$ since $i \not\sim i^0$ for all $i \in \mathbb{I}$. Therefore after renumbering the indices we may assume that $\mathbb{I} = \mathbb{I}^- \cup \mathbb{I}^+$, where $\mathbb{I}^- = \{1, \dots, \theta_1\}$, $\mathbb{I}^+ = \{\theta_1 + 1, \dots, 2\theta_1\}$, $\theta = 2\theta_1$, and $i^0 = \theta_1 + i$ for all $i \in \mathbb{I}^-$. Moreover there are subsets $\mathcal{X}^-, \mathcal{X}^+ \subset \mathcal{X}, \mathcal{X}^+ = \{J^0 \mid J \in \mathcal{X}^-\}$ such that

$$\mathbb{I}^- = \bigcup_{J \in \mathcal{X}^-} J, \quad \mathbb{I}^+ = \bigcup_{J \in \mathcal{X}^+} J.$$

Then for all $1 \le i \le \theta_1$, $\chi_i = \chi_{\theta_1+i}^{-1}$, and $\widetilde{\ell}_{\theta_1+i,j} \neq 0$ since $i, \theta_1 + i$ are linked. Define $\mathcal{D}_{red}(\Gamma, (L_i)_{1 \le \theta_1}, (K_i)_{1 \le i \le \theta_1}, (\eta_i)_{1 \le i \le \theta})$ and $\ell = (\ell_i)_{1 \le i \le \theta_1}$ by

$$\begin{aligned} (g_1, \dots, g_{2\theta_1}) &= (L_1, \dots, L_{\theta_1}, K_1, \dots, K_{\theta_1}), \\ (\chi_1, \dots, \chi_{2\theta_1}) &= (\eta_1^{-1}, \dots, \eta_{\theta_1}^{-1}, \eta_1, \dots, \eta_{\theta_1}), \\ \ell_i &= -\lambda_{\theta_1+i,i}, \qquad 1 \le i \le \theta_1. \end{aligned}$$

Then the lemma follows from Lemma 3.4 since $\mathcal{D} = \widetilde{\mathcal{D}_{red}}$, $\widetilde{\ell} = \lambda$. $\square$

3.2. The Hopf algebra $\mathcal{U}(\mathcal{D}_{red}, \ell)$ as a quotient of a Drinfeld double.

For the rest of this section we fix a reduced YD-datum

$$\mathcal{D}_{red} = \mathcal{D}(\Gamma, (L_i)_{1 \le i \le \theta}, (K_i)_{1 \le i \le \theta}, (\chi_i)_{1 \le i \le \theta})$$

with linking parameter $\ell = (\ell_i)_{1 \le i \le \theta}$, and denote $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, \ell)$. We shall describe $\mathbf{U}$ as a quotient of a Drinfeld double.

The images of v_i and w_i in $\mathbf{U}$ will again be denoted by v_i and w_i. Let

$$E_i = v_i, \quad F_i = w_i L_i^{-1} \text{ in } \mathbf{U}, \quad 1 \le i \le \theta, \tag{3.13}$$

and let $\mathbf{U}^-$ (resp. $\mathbf{U}^+$) be the subalgebra of $\mathbf{U}$ generated by $F_1, \dots, F_\theta$ (resp. $E_1, \dots, E_\theta$). Then

$$gE_ig^{-1} = \chi_i(g)E_i, \tag{3.14}$$

$$gF_ig^{-1} = \chi_i^{-1}(g)F_i, \tag{3.15}$$

$$E_iF_i - F_iE_i = \delta_{ij}\ell_i(K_i - L_i^{-1}), \tag{3.16}$$

$$\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \tag{3.17}$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i^{-1} \tag{3.18}$$

in $\mathbf{U}$, for all $1 \le i \le \theta$, $g \in \Gamma$.

Remark 3.6. The Hopf algebra $\mathcal{U}(\mathcal{D}_{red}, \ell)$ does not depend on the choice of the non-zero scalars ℓ_i. By rescaling the variables v_i we could assume that $\ell_i = 1$ for all i. By the same reason we could assume that the only values λ_{ij} of a linking parameter for a YD-datum $\mathcal{D}$ are 0 or 1.

Since $\mathcal{S}(F_i) = -F_iL_i = -w_i$ in $\mathcal{B}(W)\#\Bbbk\Gamma$ for all $1 \le i \le \theta$, the relations of the elements w_i in $\mathcal{U}(\mathcal{D}_{red}, \ell)$ may be equivalently expressed by the following relations in the F_i. If f is an element of the free algebra in the variables $x_1, \dots, x_\theta$, and the relation $f(w_1, \dots, w_\theta) = 0$ holds in $\mathcal{U}(\mathcal{D}_{red}, \ell)$, then $\widetilde{f}(F_1, \dots, F_\theta) = 0$, where

$$\Bbbk\langle x_1, \dots, x_\theta\rangle \to \Bbbk\langle x_1, \dots, x_\theta\rangle, \quad f \mapsto \widetilde{f},$$

is the vector space isomorphism mapping a monomial $x_{i_1}x_{i_2}\cdots x_{i_n}$ onto $(-1)^n x_{i_n}\cdots x_{i_2}x_{i_1}$.

Let Λ be the free abelian group with basis $z_1, \dots, z_\theta$, and define characters $\eta_j \in \widehat{\Lambda}$ by $\eta_j(z_i) = \chi_j^{-1}(L_i)$, $1 \le i, j \le \theta$. Then W is Yetter-Drinfeld module in ${}^{\Lambda}_{\Lambda}\mathcal{YD}$ with $w_i \in W^{\eta_i}_{z_i}$, $1 \le i \le \theta$, and W has the same braiding as an object in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ or in ${}^{\Lambda}_{\Lambda}\mathcal{YD}$. Hence $\mathcal{B}(W)$ is a Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ and ${}^{\Lambda}_{\Lambda}\mathcal{YD}$. Let $A = \mathfrak{B}(V)\#k[\Gamma]$ and $U = \mathfrak{B}(W)\#k[\Lambda]$.

Generally for Hopf algebras A and U a linear map $\tau : U \otimes A \to \Bbbk$ is a *skew-pairing* [DT, Definition 1.3] if

$$\tau(u, aa') = \tau(u_{(2)}, a)\tau(u_{(1)}, a'), \tag{3.19}$$

$$\tau(uu', a) = \tau(u, a_{(1)})\tau(u', a_{(2)}), \tag{3.20}$$

$$\tau(1, a) = \varepsilon(a),\ \tau(u, 1) = \varepsilon(u), \tag{3.21}$$

for all $u, u' \in U$ and $a, a' \in A$.

A skew-pairing τ defines a 2-cocycle $\sigma : (U \otimes A) \otimes (U \otimes A) \to k$ by

$$\sigma(u \otimes a, u' \otimes a') = \varepsilon(u)\tau(u', a)\varepsilon(a') \tag{3.22}$$

for all $u, u' \in U, a, a' \in A$. Let $(U \otimes A)_\sigma$ be the 2-cocycle twist of the tensor product Hopf algebra $U \otimes A$. Thus $(U \otimes A)_\sigma$ coincides with $U \otimes A$ as a coalgebra with componentwise comultiplication and its algebra structure is defined by

$$\begin{aligned}(u\otimes a)(u'\otimes a') &= \sigma(h_{(1)}, {h'}_{(1)})h_{(2)}{h'}_{(2)}\sigma^{-1}(h_{(3)}, {h'}_{(3)}) \\ &= u\tau(u'_{(1)}, a_{(1)})u'_{(2)}\otimes a_{(2)}\tau^{-1}(u'_{(3)}, a_{(3)})a'\end{aligned} \tag{3.23}$$

for all $u, u' \in U, a, a' \in A$. Note that

$$\tau^{-1}(u, a) = \tau(\mathcal{S}(u), a) = \tau(u, \mathcal{S}^{-1}(a))$$

for all $u \in U, a \in A$.

Part (1) of the next result is a special case of [RS1, Theorem 8.3, Corollary 9.1], part (2) is shown in [RS2, Theorem 4.4] for data of (finite) Cartan type, and in general by similar methods in [Ma, Theorem 5.3]. Let A, U be the bosonizations defined above.

Theorem 3.7. (1) *There is a unique skew-pairing $\tau : U \otimes A \to k$ with*

$$\tau(z_i, g) = \chi_i^{-1}(g), \qquad \tau(z_i, v_j) = 0,$$
$$\tau(w_i, g) = 0, \qquad \tau(w_i, v_j) = -\delta_{ij}\ell_i$$

for all $1 \leq i, j \leq \theta$ *and* $g \in \Gamma$.

(2) *Let* σ *be the 2-cocycle corresponding to* τ *by* (3.22). *Then there is an isomorphism of Hopf algebras*

$$\mathbf{U} \cong (U \otimes A)_\sigma / (z_i \otimes L_i^{-1} - 1 \otimes 1 \mid 1 \leq i \leq \theta),$$

mapping $w_i, 1 \leq i \leq \theta$, *and* $v_j, 1 \leq j \leq \theta$, *respectively* $g \in \Gamma$ *onto the residue classes of* $w_i \otimes 1, 1 \otimes v_j$, *respectively* $1 \otimes g$. □

The following decomposition result is a special case of [Ma, Theorem 5.2]. By definition of $\mathbf{U}$ there are algebra maps $\rho_V : \mathcal{B}(V) \to \mathbf{U}$, $\rho_W : \mathcal{B}(W) \to \mathbf{U}$ and $\rho_\Gamma : \Bbbk\Gamma \to \mathbf{U}$, given by $\rho_V(v_i) = v_i$, $\rho_W(w_i) = w_i$, $\rho_\Gamma(g) = g$, for all $1 \leq i \leq \theta, g \in \Gamma$. Clearly, the image of ρ_V coincides with $\mathbf{U}^+$– but the image of ρ_W is not $\mathbf{U}^-$.

Corollary 3.8. (1) *The multiplication map*

$$\mathcal{B}(V) \otimes \mathcal{B}(W) \otimes \Bbbk\Gamma \to \mathbf{U},\ v \otimes w \otimes g \mapsto \rho_V(v)\rho_W(w)\rho_\Gamma(g),$$

is a coalgebra isomorphism.

(2) *The multiplication map* $\mathbf{U}^- \otimes \mathbf{U}^+ \otimes k[\Gamma] \to \mathbf{U}$ *is an isomorphism of vector spaces.*

Proof. (1) follows from Theorem 1.11 and Lemma 3.5. We prove (2). By (1) we may identify $\mathcal{B}(V), \mathcal{B}(W)$ and $\Bbbk\Gamma$ with subalgebras of $\mathbf{U}$. We first claim that the multiplication map $\Bbbk\Gamma \otimes \mathcal{B}(W) \to \mathbf{U}$ defines an isomorphism

$$\Bbbk\Gamma \otimes \mathcal{B}(W) \cong \Bbbk\Gamma\mathcal{B}(W) = \mathcal{B}(W)\Bbbk\Gamma.$$

The multiplication map defines an isomorphism $\mathcal{B}(W) \otimes \Bbbk\Gamma \cong \mathcal{B}(W)\Bbbk\Gamma$ by (1). Since $gw_i = \chi^{-1}(g)w_i g$ for all $1 \leq i \leq \theta, g \in \Gamma$, $\mathcal{B}(W)$ has a vector space basis $(w_b)_{b\in B}$ such that $gw_b = \chi_b(g)w_b g$ for all $b \in B, g \in \Gamma$, where the χ_b are characters of Γ. Hence also $\Bbbk\Gamma \otimes \mathcal{B}(W) \to \mathbf{U}$ is injective, and the claim follows. Then it follows from (1) that the multiplication map

$$\mathcal{B}(V) \otimes \Bbbk\Gamma \otimes \mathcal{B}(W) \to \mathbf{U} \tag{3.24}$$

is bijective. By (1), $\mathcal{B}(V) = \mathbf{U}^+$, and $\mathcal{B}(V)\#\Bbbk\Gamma \cong \mathbf{U}^+\Bbbk\Gamma$ is a Hopf subalgebra of $\mathbf{U}$. Also, $\mathcal{S}(F_i) = -w_i$ for all $1 \leq i \leq \theta$, and $\mathcal{S}(\mathbf{U}^-) = \mathcal{B}(W)$. By (3.24) the composition

$$\mathbf{U}^+ \otimes \Bbbk\Gamma \otimes \mathbf{U}^- \cong \mathbf{U}^+\Bbbk\Gamma \otimes \mathbf{U}^- \xrightarrow{\mathcal{S}\otimes\mathcal{S}} \mathbf{U}^+\Bbbk\Gamma \otimes \mathcal{B}(W) \cong \mathbf{U},$$

mapping $x \otimes g \otimes y$ onto $\mathcal{S}(yxg)$ for all $x \in \mathbf{U}^+, g \in \Gamma, y \in \mathbf{U}^-$ is bijective. Thus multiplication defines an isomorphism $\mathbf{U}^- \otimes \mathbf{U}^+ \otimes \Bbbk\Gamma \to \mathbf{U}$. □

3.3. **A bilinear form.**

We will now see that the form $\tau : U \otimes A \to \Bbbk$ defines in a natural way a form $(\ ,\ ) : \mathbf{U}^- \otimes \mathbf{U}^+ \to \Bbbk$. This is the form we will use later on. It plays the same role as Lusztig's form $(\ ,\ ) : \mathbf{f}\otimes\mathbf{f} \to \mathbb{Q}(v)$.

Let $\pi_\Gamma : \mathfrak{B}(V)\#k[\Gamma] \to k[\Gamma]$ be the projection defined by $\pi_\Gamma(x\otimes g) = \varepsilon(x)g$ for $x \in \mathcal{B}(V)$, $g \in \Gamma$. Clearly, $\mathcal{B}(V) = A^{\operatorname{co}\pi_\Gamma}$. Let $\pi_\Lambda : \mathfrak{B}(W)\#k[\Lambda] \to k[\Lambda]$ be the analogous projection of U to $k[\Lambda]$. We have

$$\mathcal{S}(\mathcal{B}(W)) = \mathcal{S}(U^{\operatorname{co}\pi_\Lambda}) = {}^{\operatorname{co}\pi_\Lambda}U,$$

see Subsection 1.2; thus ${}^{\operatorname{co}\pi_\Lambda}U$ is generated as an algebra by the elements $w_1z_1^{-1}, \dots, w_\theta z_\theta^{-1}$ since $\mathcal{S}(w_i) = -z_i^{-1}w_i = -q_{ii}w_iz_i^{-1}$ for all $1 \le i \le \theta$.

Corollary 3.9. (1) *The Hopf algebra map* $\varphi^+ : \mathcal{B}(V)\#\Bbbk\Gamma \to \mathbf{U}$ *given by*

$$\varphi^+(v_i) = E_i, \quad \varphi^+(g) = g, \quad 1 \le i \le \theta,\ g \in \Gamma,$$

is injective. In particular,

$$\iota^+ : \mathbf{U}^+ \to A^{\operatorname{co}\pi_\Gamma},\ \iota^+(E_i) = v_i, \quad 1 \le i \le \theta$$

is a well-defined algebra isomorphism.

(2) *The Hopf algebra map* $\varphi^- : \mathcal{B}(W)\#\Bbbk\Lambda \to \mathbf{U}$ *given by*

$$\varphi^-(w_i) = F_iL_i, \quad \varphi^-(z_i) = L_i, \quad 1 \le i \le \theta,$$

induces a bijection between the subalgebras ${}^{\operatorname{co}\pi_\Lambda}U$ *and* $\mathbf{U}^-$*. In particular,*

$$\iota^- : \mathbf{U}^- \to {}^{\operatorname{co}\pi_\Lambda}U, \quad \iota^-(F_i) = w_iz_i^{-1}, 1 \le i \le \theta,$$

is a well-defined algebra isomorphism.

(3) *Let* κ *be the bijective map of Lemma 1.3 with respect to the projection* $\pi = \pi_\Lambda : \mathcal{B}(W)\#\Bbbk\Lambda \to \Bbbk\Lambda$*. Then* $\varphi^-\kappa$ *defines a bijective linear map*

$$\widetilde{\kappa} : \mathcal{B}(W) \to \mathbf{U}^-,\ \widetilde{\kappa}(w_{i_1}\cdots w_{i_n}) = \prod_{l=1}^{n} q_{i_li_l} \prod_{k>l} q_{i_ki_l}^{-1} F_{i_1}\cdots F_{i_n}$$

for all $1 \le i_1, \dots, i_n \le \theta, n \ge 1$.

Proof. (1) follows from Corollary 3.8 (1).

(2) The Hopf algebra map φ^- is the composition of the Hopf algebra maps $\mathcal{B}(W)\#\Bbbk\Lambda \to \mathcal{B}(W)\#\Bbbk\Gamma$, $w_i \mapsto w_i$, $z_i \mapsto L_i$, $1 \le i \le \theta$ and $\mathcal{B}(W)\#\Bbbk\Gamma \to \mathbf{U}$, $w_i \mapsto w_i = F_iL_i$, $g \mapsto g$, $1 \le i \le \theta$, $g \in \Gamma$. The restriction of φ^- is an isomorphism from $\mathfrak{B}(W)$ to the subalgebra $\Bbbk\langle w_1, \dots, w_\theta\rangle$ of $\mathbf{U}$, by (1). Hence φ^- induces an isomorphism between $\mathcal{S}(\mathfrak{B}(W)) = {}^{\operatorname{co}\pi_\Lambda}U$ and $\mathcal{S}(\Bbbk\langle w_1, \dots, w_\theta\rangle) = \mathbf{U}^-$. Its inverse is ι^-.

(3) follows from (2) and the formula for κ in Lemma 1.3. □

Definition 3.10. The $\Bbbk$-bilinear form $(\ ,\ ) : \mathbf{U}^- \otimes \mathbf{U}^+ \to \Bbbk$ is defined by $(x, y) = \tau(\iota^-(x), \iota^+(y))$ for all $x \in \mathbf{U}^-, y \in \mathbf{U}^+$.

If $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}^{\mathbb{I}}$, $n_1, \dots, n_\theta \in \mathbb{Z}$, we let $|\alpha| = \sum_{i=1}^{\theta} n_i$, and

$$\chi_\alpha = \chi_1^{n_1}\cdots\chi_\theta^{n_\theta}, \quad K_\alpha = K_1^{n_1}\cdots K_\theta^{n_\theta}, \quad L_\alpha = L_1^{n_1}\cdots L_\theta^{n_\theta}. \tag{3.25}$$

The Hopf algebras $U = \mathfrak{B}(W)\#k[\Lambda]$ and $A = \mathfrak{B}(V)\#k[\Gamma]$ are $\mathbb{N}^{\mathbb{I}}$-graded as algebras and coalgebras where the elements v_i have degree α_i and the elements w_i have degree $-\alpha_i$ for all $1 \le i \le \theta$, and the elements of the groups Λ and Γ have degree 0. Hence the algebras $\mathbf{U}^+, \mathbf{U}^-$ are $\mathbb{N}^{\mathbb{I}}$-graded

by Corollary 3.9, where the degree of E_i is α_i and the degree of F_i is $-\alpha_i$, for all $1 \le i \le \theta$.

We collect some important properties of the forms τ and $(\ , \ )$.

Theorem 3.11. *(1) $\tau(uz, ag) = \tau(u, a)\tau(z, g)$, for all $u \in {}^{\operatorname{co}\pi_\Lambda}U$, $a \in A^{\operatorname{co}\pi_\Gamma} = \mathfrak{B}(V)$, $z \in \Lambda, g \in \Gamma$.*

(2) For all $\alpha, \beta \in \mathbb{N}^{\mathbb{I}}$, $\alpha \neq \beta$, $\tau(U_{-\alpha}, A_\beta) = 0$.

(3) For all $\alpha \in \mathbb{N}^{\mathbb{I}}$, the restriction of τ to $\mathfrak{B}(W)_{-\alpha} \times \mathfrak{B}(V)_\alpha$ is non-degenerate.

(4) For all $\alpha, \beta \in \mathbb{N}^{\mathbb{I}}, \alpha \neq \beta$, $(\mathbf{U}^-{}_{-\alpha}, \mathbf{U}^+{}_\beta) = 0$.

(5) For all $\alpha \in \mathbb{N}^{\mathbb{I}}$, the restriction of the form $(\ , \ )$ to $\mathbf{U}^-_{-\alpha} \times \mathbf{U}^+_\alpha$ is non-degenerate.

(6) For all $1 \le i, j \le \theta, (F_i, E_j) = -\delta_{ij}\ell_i$.

Proof. (1) The proof follows from the claims

(a) $\tau(z, a) = \tau(z, \pi_\Gamma(a))$,

(b) $\tau(u, g) = \tau(\pi_\Lambda(u), g)$

for all $z \in \Lambda$, $a \in A$, $u \in U$, $g \in \Gamma$. For, suppose (a) and (b) hold and u, a, z, g satisfy the hypothesis of (1). Then using Theorem 3.7 we calculate

$$\begin{aligned}
\tau(uz, ag) &= \tau(u, a_{(1)}g)\tau(z, a_{(2)}g) \\
&= \tau(u_{(2)}, a_{(1)})\tau(u_{(1)}, g)\tau(z, a_{(2)})\tau(z, g) \\
&= \tau(u_{(2)}, a_{(1)})\tau(\pi_\Lambda(u_{(1)}), g)\tau(z, \pi_\Gamma(a_{(2)}))\tau(z, g) \\
&= \tau(u, a)\tau(z, g).
\end{aligned}$$

We prove (a). Since $z \in G(U)$, the map $\tau(z, -) : A \to k$ is an algebra map by part (1) of Theorem 3.7. Since $K_i v_i K_i^{-1} = q_{ii} v_i$ and $q_{ii} \neq 1$ for all $1 \le i \le \theta$ it follows that any algebra map from A to k vanishes on each v_i. Thus $\tau(z, a) = \tau(z, \pi_\Gamma(a))$ for all $a \in A$. The second claim (b) follows similarly using $\tau(-, g)$ in place of $\tau(z, -)$.

(2) follows from Theorem 3.7 (1) and the fact that the comultiplications of U and A are $\mathbb{Z}^{\mathbb{I}}$-graded.

(3) Since all the ℓ_i are non-zero, the form τ restricts to a non-degenerate pairing between $\mathcal{B}(W)$ and $\mathcal{B}(V)$ (see [RS1] or [RS2, Remark 3.3]). Hence the claim in (3) follows from (2).

(4) and (5) follow from (2) and (3) using (1), since $U = {}^{\operatorname{co}\pi_\Lambda}U\Lambda$.

(6) follows from Theorem 3.7 (1). □

3.4. **Further properties of the bilinear form.**

We now discuss some further properties of the bilinear form following [L2, Chapters 3 and 4]; in particular, we study a universal element in some completion of $\mathbf{U}$. In the case of reduced data of Cartan type, it will give rise to Casimir elements, up to some suitable modification.

In [L2, 1.2.13] Lusztig introduces two skew-derivations r_i and ${}_ir$. We need four such maps. The comultiplication of $\mathbf{U}$ defines skew-derivations

$r_i, r'_i : \mathbf{U}^+ \to \mathbf{U}^+$ and $s_i, s'_i : \mathbf{U}^- \to \mathbf{U}^-$ for all $1 \leq i \leq \theta$ in the following way.

Since $\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1$, for all $1 \leq i \leq \theta$, it follows that for all $\alpha \in \mathbb{N}^{\mathbb{I}}$ and $y \in \mathbf{U}^+_\alpha$, $\Delta(y)$ has the form

$$\Delta(y) = y \otimes 1 + \sum_{i=1}^{\theta} r_i(y) K_i \otimes E_i + \text{ terms of other degrees,} \tag{3.26}$$

$$\Delta(y) = K_\alpha \otimes y + \sum_{i=1}^{\theta} E_i K_{\alpha-\alpha_i} \otimes r'_i(y) + \text{ terms of other degrees,} \tag{3.27}$$

where $r_i(y), r'_i(y)$ are uniquely determined elements in $\mathbf{U}^+_{\alpha-\alpha_i}$. Degree refers to the standard $\mathbb{Z}^{\mathbb{I}}$-grading in the tensor product. Then for all $y, y' \in \mathbf{U}^+$ and $1 \leq i \leq \theta$,

$$r_i(yy') = y r_i(y') + r_i(y)(K_i \triangleright y'), \tag{3.28}$$

$$r'_i(yy') = (L_i \triangleright y) r'_i(y') + r'_i(y) y'. \tag{3.29}$$

This follows from $\Delta(yy') = \Delta(y)\Delta(y')$ by comparing coefficients. Note that $r_i(E_j) = \delta_{ij}$, $r'_{ij}(E_j) = \delta_{ij}$ for all $1 \leq i, j \leq \theta$.

In the same way it follows from $\Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i^{-1}$ for all $1 \leq i \leq \theta$ that for all $\alpha \in \mathbb{N}^{\mathbb{I}}$ and $x \in \mathbf{U}^-_{-\alpha}$,

$$\Delta(x) = x \otimes L_\alpha^{-1} + \sum_{i=1}^{\theta} s_i(x) \otimes F_i L^{-1}_{\alpha-\alpha_i} + \text{ terms of other degrees,} \tag{3.30}$$

$$\Delta(x) = 1 \otimes x + \sum_{i=1}^{\theta} F_i \otimes s'_i(x) L_i^{-1} + \text{ terms of other degrees,} \tag{3.31}$$

where $s_i(x)$, $s'_i(x) \in \mathbf{U}^-_{-\alpha+\alpha_i}$ are uniquely determined elements. Then for all $1 \leq i, j \leq \theta$, $s_i(F_j) = \delta_{ij}$, $s'_{ij}(F_j) = \delta_{ij}$, and for all $x, x' \in \mathbf{U}^-$ and $1 \leq i \leq \theta$,

$$s_i(xx') = (K_i^{-1} \triangleright x) s_i(x') + s_i(x) x', \tag{3.32}$$

$$s'_i(xx') = x s'_i(x') + s'_i(x)(L_i^{-1} \triangleright x'). \tag{3.33}$$

The next Propositions 3.12, 3.13 extend [L2, 3.1.6].

Proposition 3.12. *For all* $x \in \mathbf{U}^-, y \in \mathbf{U}^+$ *and* $1 \leq i \leq \theta$,

(1) $$yF_i - F_i y = \ell_i(r_i(y) K_i - L_i^{-1} r'_i(y)),$$

(2) $$(xF_i, y) = (x, r_i(y))(F_i, E_i),$$

(3) $$(F_i x, y) = (x, r'_i(y))(F_i, E_i).$$

Proof. (1) The function $d_i : \mathbf{U}^+ \to \mathbf{U},\ y \mapsto r_i(y)K_i - L_i^{-1}r_i'(y)$, is a derivation since for all $y, y' \in \mathbf{U}^+$,

$$\begin{aligned} d_i(yy') &= r_i(yy')K_i - L_i^{-1}r_i'(yy') \\ &= yr_i(y')K_i + r_i(y)(K_i \triangleright y')K_i - L_i^{-1}(L_i \triangleright y)r_i'(y') - L_i^{-1}r_i'(y)y' \\ &= yr_i(y')K_i + r_i(y)K_iy' - yL_i^{-1}r_i'(y') - L_i^{-1}r_i'(y)y' \\ &= d_i(y)y' + yd_i(y'), \end{aligned}$$

where we have used (3.28), (3.29) and the equalities

$$(K_i \triangleright y')K_i = K_iy', \quad L_i^{-1}(L_i \triangleright y) = yL_i^{-1}.$$

Moreover, $d_i(E_j) = \delta_{ij}(K_i - L_i^{-1})$, for all $1 \le j \le \theta$. Since both sides of (1) are derivations having the same values on the generators E_j of $\mathbf{U}^+$, the claim follows.

(2) We can assume that $y \in \mathbf{U}^+_\alpha$, where $\alpha \in \mathbb{N}^{\mathbb{I}}$. Let $u = \iota^-(x)$, $a = \iota^+(y)$. Then

$$\Delta(a) = a \otimes 1 + \sum_{i=1}^{\theta} \bar{r}_i(a)K_i \otimes v_i + \text{ terms of other degrees,}$$

where $\bar{r}_i(a) = \iota^+(r_i(y))$ by Corollary 3.9 (1). Hence, by Lemmas 3.7, 3.11,

$$\begin{aligned} (xF_i, y) &= \tau(uw_iz_i^{-1}, a) \\ &= \tau(u, a_{(1)})\tau(w_iz_i^{-1}, a_{(2)}) \\ &= \tau(u, \bar{r}_i(a)K_i)\tau(w_iz_i^{-1}, v_i) \\ &= \tau(u, \bar{r}_i(a))\tau(w_iz_i^{-1}, v_i) \\ &= (x, r_i(y))(F_i, E_i). \end{aligned}$$

(3) is proved in the same way as (2). □

Proposition 3.13. *For all $x \in \mathbf{U}^-, y \in \mathbf{U}^+$ and $1 \le i \le \theta$,*

(1) $$E_ix - xE_i = \ell_i(K_is_i(x) - s_i'(x)L_i^{-1}),$$

(2) $$(x, E_iy) = (s_i(x), y)(F_i, E_i),$$

(3) $$(x, yE_i) = (s_i'(x), y)(F_i, E_i).$$

Proof. Similar to the proof of Proposition 3.12 using Corollary 3.9 (2). □

Recall that the form $(\ ,\ ) : \mathbf{U}^-_{-\alpha} \times \mathbf{U}^+_\alpha \to k$ is non-degenerate by Theorem 3.11 (5), for all $\alpha \in \mathbb{N}^{\mathbb{I}}$.

Definition 3.14. For all $\alpha \in \mathbb{N}^{\mathbb{I}}$, let x^k_α, $1 \le k \le d_\alpha = \dim \mathbf{U}^-_{-\alpha}$, be a basis of $\mathbf{U}^-_{-\alpha}$; and $y^k_\alpha, 1 \le k \le d_\alpha$, the dual basis of $\mathbf{U}^+_\alpha$ with respect to $(\ ,\ )$. Define

$$\theta_\alpha = \sum_{k=1}^{d_\alpha} x^k_\alpha \otimes y^k_\alpha.$$

We set $\theta_\alpha = 0$ for all $\alpha \in \mathbb{Z}^{\mathbb{I}}$ and $\alpha \notin \mathbb{N}^{\mathbb{I}}$. The following formal element is instrumental to the definition of the quantum Casimir element:

$$\Omega = \sum_{\alpha \in \mathbb{N}^{\mathbb{I}}} \sum_{k=1}^{d_\alpha} \mathcal{S}(x_\alpha^k) y_\alpha^k.$$

We collect some general properties of the family (θ_α) generalizing [L2, 4.2.5].

Theorem 3.15. *Let $\alpha \in \mathbb{N}^{\mathbb{I}}$ and $1 \leq i \leq \theta$. Then in $\mathbf{U} \otimes \mathbf{U}$,*

$$(E_i \otimes 1)\theta_\alpha + (K_i \otimes E_i)\theta_{\alpha-\alpha_i} = \theta_\alpha(E_i \otimes 1) + \theta_{\alpha-\alpha_i}(L_i^{-1} \otimes E_i), \tag{1}$$

$$(1 \otimes F_i)\theta_\alpha + (F_i \otimes L_i^{-1})\theta_{\alpha-\alpha_i} = \theta_\alpha(1 \otimes F_i) + \theta_{\alpha-\alpha_i}(F_i \otimes K_i). \tag{2}$$

Proof. Both equalities hold when $\alpha - \alpha_i \notin \mathbb{N}^{\mathbb{I}}$ since then E_i commutes with the elements x_α^k which are products of $F_j's$ where $j \neq i$, and similarly F_i commutes with the elements y_α^k.

By definition the equality in (1) means that

$$\begin{aligned} &\sum_k E_i x_\alpha^k \otimes y_\alpha^k + \sum_l K_i x_{\alpha-\alpha_i}^l \otimes E_i y_{\alpha-\alpha_i}^l \\ &- \sum_k x_\alpha^k E_i \otimes y_\alpha^k - \sum_l x_{\alpha-\alpha_i}^l L_i^{-1} \otimes y_{\alpha-\alpha_i}^l E_i = 0 \end{aligned}$$

in $\mathbf{U} \otimes \mathbf{U}_\alpha^+$, or equivalently, by non-degeneracy of $(\ , \ )$, that

$$\begin{aligned} &\sum_k (E_i x_\alpha^k - x_\alpha^k E_i)(z, y_\alpha^k) + \sum_l K_i x_{\alpha-\alpha_i}^l (z, E_i y_{\alpha-\alpha_i}^l) \\ &- \sum_l x_{\alpha-\alpha_i}^l L_i^{-1}(z, y_{\alpha-\alpha_i}^l E_i) = 0, \end{aligned}$$

for all $z \in \mathbf{U}_{-\alpha}^-$. Now we apply Proposition 3.13 (1), (2) and (3) to the summands of the first, second and third sum, collect coefficients of K_i and L_i^{-1} and obtain the following equivalent form of (1)

$$\begin{aligned} &K_i \left((F_i, E_i) \sum_l x_{\alpha-\alpha_i}^l (s_i(z), y_{\alpha-\alpha_i}^l) + \ell_i \sum_k s_i(x_\alpha^k)(z, y_\alpha^k) \right) \\ &- \left(\ell_i \sum_k s_i'(x_\alpha^k)(z, y_\alpha^k) + (F_i, E_i) \sum_l x_{\alpha-\alpha_i}^l (s_i'(z), y_{\alpha-\alpha_i}^l) \right) L_i^{-1} = 0. \end{aligned}$$

Since the tensorands of θ_α and $\theta_{\alpha-\alpha_i}$ are dual bases, we see that

$$\begin{aligned} &\sum_l x_{\alpha-\alpha_i}^l (s_i(z), y_{\alpha-\alpha_i}^l) = s_i(z), \\ &\sum_k s_i(x_\alpha^k)(z, y_\alpha^k)) = s_i(\sum_k x_\alpha^k (z, y_\alpha^k)) = s_i(z). \end{aligned}$$

Since $(F_i, E_i) = -\ell_i$, it follows that the coefficient of K_i is zero. Similarly the coefficient of L_i^{-1} is zero since

$$\sum_k s_i'(x_\alpha^k)(z, y_\alpha^k) = s_i'(z),$$
$$\sum_l x_{\alpha-\alpha_i}^l (s_i'(z), y_{\alpha-\alpha_i}^l) = s_i'(z).$$

(2) is proved in the same way using Proposition 3.12 instead of Proposition 3.13. □

3.5. **Data of Cartan type.**

Let $(a_{ij})_{1\le i,j\le\theta}$ be a generalized Cartan matrix, that is, $(a_{ij})_{1\le i,j\le\theta}$ is a matrix with has integer entries such $a_{ii} = 2$ for all $1 \le i \le \theta$, and for all $1 \le i, j \le \theta, i \ne j$, $a_{ij} \le 0$, and if $a_{ij} = 0$, then $a_{ji} = 0$.

Definition 3.16. Let $\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$ be a YD-datum.

We say that $\mathcal{D}$ is a *YD-datum of Cartan type* (a_{ij}) if

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \ne 1,\ 0 \le -a_{ij} < \operatorname{ord}(q_{ii}),\ \text{for all } 1 \le i, j \le \theta, \tag{3.34}$$

where the q_{ij} are defined by (1.1), and $1 \le \operatorname{ord}(q_{ii}) \le \infty$.

Note that the equivalence relation (1.2) can be described as usual in terms of the Cartan matrix. For all $1 \le i, j \le \theta$, $i \sim j$ if and only if there are vertices $i_1, \dots, i_t \in \mathbb{I}, t \ge 2$ with $i_1 = i$, $i_t = j$, $a_{i_l, i_{l+1}} \ne 0$ for all $1 \le l < t$.

Definition 3.17. A *reduced YD-datum of Cartan type*

$$\mathcal{D}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta})$$

is a reduced YD-datum $\mathcal{D}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$ such that for all $1 \le i, j \le \theta$

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad q_{ii} \ne 1, \quad 0 \le -a_{ij} < \operatorname{ord}(q_{ii}), \tag{3.35}$$

where $q_{ij} = \chi_j(K_i)$, as in page 13.

We introduce an important condition which generalizes the notion of X-regular root data in [L2, Chapter 2].

Definition 3.18. A reduced YD-datum

$$\mathcal{D}_{red} = \mathcal{D}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta})$$

is called *regular* if the characters $\chi_1, \dots, \chi_\theta$ are $\mathbb{Z}$-linearly independent in $\widehat{\Gamma}$.

We fix a generic, see Definition 1.4, reduced YD-datum of Cartan type

$$\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i)_{1\le i\le\theta}, (K_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\theta}).$$

By [AS3, Lemma 2.4] we can choose $d_1, \dots, d_\theta \in \mathbb{N} - 0$ such that

$$d_i a_{ij} = d_j a_{ji} \text{ for all } i, j \in \mathbb{I}. \tag{3.36}$$

Let $\mathcal{X}$ be the set of connected components of $\mathbb{I} = \{1, \dots, \theta\}$ with respect to the Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$.

It is useful to single out the following subgroup of Γ.

Definition 3.19. Let Γ^2 be the subgroup of Γ generated by the products $K_1L_1, \dots, K_\theta L_\theta$.

Lemma 3.20. (1) *Let $J \subset \mathbb{I}$ be a connected component. Then there are $q_J \in \Bbbk^\times$ which is not a root of unity, and roots of unity $\zeta_j \in \Bbbk$, $j \in J$, such that $q_{jj} = q_J^{2d_j}\zeta_j$ for all $j \in J$. In particular, the elements $(q_{jj})_{j\in J}$ are $\mathbb{N}$-linearly independent, that is, if $(n_j)_{j\in J}$ is a family of natural numbers, then $\prod_{j\in J} q_{jj}^{n_j} = 1$ implies that $n_j = 0$ for all $j \in J$.*

(2) *If (a_{ij}) is invertible,* e. g. *if it is a Cartan matrix of finite type, then $\mathcal{D}_{red}$ is regular.*

(3) *If $\mathcal{D}_{red}$ is regular and the index of Γ^2 in Γ is finite, then the Cartan matrix (a_{ij}) is invertible.*

(4) *If (a_{ij}) is a Cartan matrix of finite type, then for all connected components $J \subset \mathbb{I}$ there is an element $q_J \in \Bbbk^\times$ such that*

$$q_{ii} = q_J^{d_i a_{ij}} \text{ for all } i \in J,\ J \in \mathcal{X}. \tag{3.37}$$

Proof. (1) We choose an element $i \in J$, and $q_J \in \Bbbk$ with $q_{ii} = q_J^{2d_i}$. Then for all $j \in J$ there are $i_1, \dots, i_t \in J$, $t \geq 2$, with $i_1 = i$, $i_t = j$, and $a_{i_l i_{l+1}} \neq 0$ for all $1 \leq l < t$. By applying (3.35) several times we obtain

$$q_{ii}^{a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{t-1}i_t}} = q_{jj}^{a_{i_2i_1}a_{i_3i_2}\cdots a_{i_ti_{t-1}}}.$$

On the other hand

$$q_{ii}^{a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{t-1}i_t}} = q_J^{2d_ia_{i_1i_2}a_{i_2i_3}\cdots a_{i_{t-1}i_t}} = q_J^{2d_ja_{i_2i_1}a_{i_3i_2}\cdots a_{i_ti_{t-1}}}$$

by applying (3.36) several times. Hence for all $j \in J$ there is a root of unity $\zeta_j \in \Bbbk$ such that $q_{jj} = q_J^{2d_j}\zeta_j$. In particular, the elements $(q_{jj})_{j\in J}$ are $\mathbb{N}$-linearly independent since $\mathcal{D}_{red}$ is generic, hence q_J is not a root of unity.

(2) Suppose $n_1, \dots, n_\theta$ are integers with $\prod_{i=1}^{\theta} \chi_i^{n_i} = 1$. Let $J \subset \mathbb{I}$ be a connected component. Since $\chi_i(K_jL_j) = q_{ii}{}^{a_{ij}}$ for all i, j by (3.35), (3.1) we obtain for all $j \in J$

$$1 = \prod_{i=1}^{\theta} \chi_i^{n_i}(K_jL_j) = \prod_{i\in J} q_{ii}^{n_ia_{ij}},$$

where in the last product we can assume that $i \in J$ since $a_{ij} = 0$ for all $i \notin J$. By the proof of (1) we may write $q_{ii} = q_J^{2d_i}\zeta_i$ for all $i \in J$, where the ζ_i are roots of unity. Thus $\prod_{i\in J} q_J^{2d_in_ia_{ij}} = 1$ for all $j \in J$ since q_J is not a root of unity. Since $(a_{ij})_{i,j\in J}$ is invertible, it follows that $n_i = 0$ for all $i \in J$.

(3) Since $\chi_1, \dots, \chi_\theta$ are $\mathbb{Z}$-linearly independent characters and Γ/Γ^2 is a finite group, the restrictions of $\chi_1, \dots, \chi_\theta$ to the subgroup Γ^2 are $\mathbb{Z}$-linearly

independent. Let J be any connected component of I with respect to (a_{ij}). We use the notation of the first part of the proof. Then $\chi_j(K_iL_i) = q_{ii}^{a_{ij}} = q_J^{2d_ia_{ij}}\zeta_i$ for all $i,j \in J$. Assume $n_j, j \in J$ are integers with $\sum_{j\in J} a_{ij}n_j = 0$ for all $i \in J$. Let $n \in \mathbb{N}$ with $\zeta_i^n = 1$ for all $i \in J$. Then

$$\prod_{j\in J} \chi_j^{nn_j}(K_iL_i) = \prod_{j\in J}(q_J^{2d_ia_{ij}nn_j}\zeta_i^{nn_j}) = q_J^{\sum_{j\in J} 2d_ia_{ij}nn_j} = 1$$

for all $i \in J$. Since the restrictions of the characters $\chi_j, j \in J$ to Γ^2 are $\mathbb{Z}$-linearly independent, and since $\chi_j(K_iL_i) = 1$ for all $j \in J, i \in I \setminus J$, it follows that $n_j = 0$ for all $j \in J$. Hence the matrix $(a_{ij})_{i,j\in J}$ is invertible. Since J was an arbitrary connected component, the claim is proved.

(4) Finally, it is not difficult to see, by inspection, that (3.37) holds for data of finite Cartan type. □

Remark 3.21. (1) The following relations hold in $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, \ell)$ for all $1 \leq i,j \leq \theta, i \neq j$:

$$\operatorname{ad}_l(E_i)^{1-a_{ij}}(E_j) = \sum_{s=0}^{1-a_{ij}} c_{ijs}E_i^sE_jE_i^{1-a_{ij}-s} = 0, \tag{3.38}$$

$$\operatorname{ad}_r(F_i)^{1-a_{ij}}(F_j) = \sum_{s=0}^{1-a_{ij}} d_{ijs}F_i^sF_jF_i^{1-a_{ij}-s} = 0, \tag{3.39}$$

where for all $1 \leq i,j \leq \theta, i \neq j, 0 \leq s \leq 1-a_{ij}$, c_{ijs}, d_{ijs} are non-zero elements in $\Bbbk$.

Proof. The first equality in (3.38) follows from the quantum binomial formula in $\operatorname{End}(\mathbf{U}^+)$, since for all $1 \leq i \leq \theta$, $\operatorname{ad}_l(E_i) = L_{E_i} - R_{E_i}\operatorname{ad}_l K_i$, and $(R_{E_i}\operatorname{ad}_l K_i)L_{E_i} = q_{ii}L_{E_i}(R_{E_i}\operatorname{ad}_l K_i)$, where $L_{E_i}(x) = L_ix$ and $R_{E_i}(x) = xE_i$ for all $x \in \mathbf{U}^+$. In same way the first equality in (3.39) is shown. The second equality in (3.38) follows from (1.5) since by definition the elements $v_i = E_i, 1 \leq i \leq \theta$, satisfy the relations of the Nichols algebra $\mathcal{B}(V)$. By the same reason $\operatorname{ad}_l(w_i)^{1-a_{ij}}(w_j) = 0$ for all $1 \leq i,j \leq \theta, i \neq j$. Hence $\mathcal{S}(\operatorname{ad}_l(w_i)^{1-a_{ij}}(w_j)) = \operatorname{ad}_r(\mathcal{S}(w_i))^{1-a_{ij}}(\mathcal{S}(w_j)) = 0$ for all $1 \leq i,j \leq \theta, i \neq j$, where $\mathcal{S}$ is the antipode of the Hopf algebra $\mathcal{B}(W)\#\Bbbk[\Lambda]$. This proves the second equality in (3.39) since by Corollary 3.9 (2) $\varphi^-(\mathcal{S}(w_i)) = -q_{ii}F_i$ for all $1 \leq i \leq \theta$. □

(2) Assume that the braiding matrix (q_{ij}) satisfies

$$q_{ii} = q_J^{2d_i} \quad \text{for all } i \in J,\ J \in \mathcal{X}. \tag{3.40}$$

Then (q_{ij}) is twist-equivalent to a braiding of Drinfeld-Jimbo type [AS3]. Indeed, let $\hat{q}_{ij} = q_J^{d_ia_{ij}}$, for all $J \in \mathcal{X}$ and $i,j \in J$; set $\hat{q}_{ij} = 1$, for $1 \leq i,j \leq \theta$ such that $i \nsim j$. Then $(\hat{q}_{ij})$ is of Drinfeld-Jimbo type, and the braidings (q_{ij}) and $(\hat{q}_{ij})$ are twist-equivalent since $q_{ij}q_{ji} = \hat{q}_{ij}\hat{q}_{ji}, q_{ii} = \hat{q}_{ii}$ for all i, j.

In this case, the braided Serre relations (1.5) are defining relations of the Nichols algebras $\mathfrak{B}(V)$ and $\mathfrak{B}(W)$. This follows by twisting from [L2, 33.1.5] when the elements $q_J \in \Bbbk$ are transcendental, and from [Ro2, Theorem 15] (see also [HK, Subsection 3.4]) when they are not roots of unity. Thus in Definition 3.3 the relations (3.5), (3.6) of $\mathbf{U}$ can be replaced by (3.38) and (3.39).

To describe the relations explicitly (cf. [RS2, Lemma 1.6]), let

$$p_{ij} = q_{ij}\hat{q}_{ij}^{-1}, \quad i \in J,\ J \in \mathcal{X},\ 1 \le j \le \theta. \tag{3.41}$$

Then (3.38), (3.39) are equivalent to

$$\sum_{s=0}^{1-a_{ij}} (-p_{ij})^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_J^{d_i}} E_i^{1-a_{ij}-s} E_j E_i^s = 0, \tag{3.42}$$

$$\sum_{s=0}^{1-a_{ij}} (-p_{ij})^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_J^{d_i}} F_i^s F_j F_i^{1-a_{ij}-s} = 0, \tag{3.43}$$

for all $i \in J$, $J \in \mathcal{X}$ and $1 \le j \le \theta$, $i \ne j$.

4. Representation theory of $\mathbf{U}$

In this section we assume that $\mathcal{D}_{red}$ is generic, regular, and of Cartan type; we denote $\mathbf{U} = \mathcal{U}(\mathcal{D}_{red}, \ell)$. We extend [L2, Sections 3.4 and 3.5].

Let Q be the subgroup of $\widehat{\Gamma}$ generated by $\chi_1, \dots, \chi_\theta$. Thus by regularity

$$\mathbb{Z}^{\mathbb{I}} \xrightarrow{\cong} Q, \quad \alpha \mapsto \chi_\alpha,$$

is bijective.

4.1. The category $\mathcal{C}^{hi}$.

Let $\mathcal{C}$ be the full subcategory of ${}_{\mathbf{U}}\mathcal{M}$ consisting of all left $\mathbf{U}$-modules M which are direct sums of 1-dimensional Γ-modules, that is, which have a weight space decomposition $M = \oplus_{\chi\in\widehat{\Gamma}} M^\chi$, where

$$M^\chi = \{m \in M \mid gm = \chi(g)m \text{ for all } g \in \Gamma\}$$

for all $\chi \in \widehat{\Gamma}$. A character $\chi \in \widehat{\Gamma}$ is called a *weight* for M if $M^\chi \ne 0$.

Let $\mathcal{C}^{hi}$ be the full subcategory of $\mathcal{C}$ defined as follows. A module $M \in \mathcal{C}$ is in $\mathcal{C}^{hi}$ if for any $m \in M$ there is an integer $N \ge 0$ such that $\mathbf{U}^+_\alpha m = 0$ for all $\alpha \in \mathbb{N}^{\mathbb{I}}$ with $|\alpha| \ge N$.

Note that both categories $\mathcal{C}$ and $\mathcal{C}^{hi}$ are closed under sub-objects and quotient objects in ${}_{\mathbf{U}}\mathcal{M}$.

We begin with a technical result to be used later.

Proposition 4.1. *Let $M \in \mathcal{C}^{hi}$. Then multiplication with Ω on M is a well-defined operator mapping each weight space of M into itself. For all $\chi \in \widehat{\Gamma}, m \in M^\chi$, and $1 \leq i \leq \theta$,*

$$\Omega E_i m = (\chi\chi_i)(K_iL_i)^{-1}E_i\Omega m, \tag{1}$$

$$\Omega F_i m = \chi(K_iL_i)F_i\Omega m. \tag{2}$$

Proof. For all $m \in M$, $\Omega m = \sum_{\alpha\in\mathbb{N}^{\mathbb{I}}}\sum_{k=1}^{d_\alpha} S(x_\alpha^k)y_\alpha^k m$ is a finite sum since $M \in \mathcal{C}^{hi}$. Hence multiplication with Ω is a well-defined operator on M. For all $\alpha \in \mathbb{N}^{\mathbb{I}}$ and $x \in \mathbf{U}^-_{-\alpha}, y \in \mathbf{U}^+_\alpha$ the element $\mathcal{S}(x)y$ commutes with all $g \in \Gamma$. Hence $\Omega : M \to M$ is Γ-linear and maps each weight space of M into itself.

To prove (1) let $\chi \in \widehat{\Gamma}$ and $m \in M^\chi$. We apply $\mathcal{S}\otimes \mathrm{id}$ to Theorem 3.15 (1), multiply and obtain for all $\alpha \in \mathbb{N}^{\mathbb{I}}$

$$\begin{aligned}&\sum_{k=1}^{d_\alpha}\mathcal{S}(x_\alpha^k)S(E_i)y_\alpha^k + \sum_{l=1}^{d_{\alpha-\alpha_i}}\mathcal{S}(x^l_{\alpha-\alpha_i})K_i^{-1}E_iy^l_{\alpha-\alpha_i}\\ &=\sum_{k=1}^{d_\alpha}\mathcal{S}(E_i)\mathcal{S}(x_\alpha^k)y_\alpha^k + \sum_{l=1}^{d_{\alpha-\alpha_i}}L_i\mathcal{S}(x^l_{\alpha-\alpha_i})y^l_{\alpha-\alpha_i}E_i.\end{aligned}$$

Here both sums over l are zero if $\alpha-\alpha_i \notin \mathbb{N}^{\mathbb{I}}$. Since $\mathcal{S}(E_i) = -K_i^{-1}E_i$ it follows that

$$\begin{aligned}&-\sum_{\alpha\in\mathbb{N}^{\mathbb{I}}}\sum_{k=1}^{d_\alpha}\mathcal{S}(x_\alpha^k)K_i^{-1}E_iy_\alpha^k m + \sum_{\alpha\in\mathbb{N}^{\mathbb{I}}}\sum_{l=1}^{d_{\alpha-\alpha_i}}\mathcal{S}(x^l_{\alpha-\alpha_i})K_i^{-1}E_iy^l_{\alpha-\alpha_i}m\\ &=-\sum_{\alpha\in\mathbb{N}^{\mathbb{I}}}\sum_{k=1}^{d_\alpha}K_i^{-1}E_i\mathcal{S}(x_\alpha^k)y_\alpha^k m + \sum_{\alpha\in\mathbb{N}^{\mathbb{I}}}\sum_{l=1}^{d_{\alpha-\alpha_i}}L_i\mathcal{S}(x^l_{\alpha-\alpha_i})y^l_{\alpha-\alpha_i}E_im.\end{aligned}$$

Since the left hand side of the last equation is zero we obtain

$$0 = -K_i^{-1}E_i\Omega m + L_i\Omega E_i m,$$

hence

$$\Omega E_i m = (K_iL_i)^{-1}E_i\Omega m = (\chi\chi_i)(K_iL_i)^{-1}E_i\Omega m.$$

In the same way (2) follows from Theorem 3.15 (2). □

Let $\chi \in \widehat{\Gamma}$. We define the *Verma module*

$$M(\chi) = \mathbf{U}/\big(\sum_{i=1}^{\theta}\mathbf{U}E_i + \sum_{g\in\Gamma}\mathbf{U}(g-\chi(g))\big).$$

The inclusion $\mathbf{U}^- \subset \mathbf{U}$ defines a $\mathbf{U}^-$-module isomorphism

$$\mathbf{U}^- \xrightarrow{\cong} M(\chi) = \mathbf{U}/\big(\sum_{i=1}^{\theta} \mathbf{U}E_i + \sum_{g\in\Gamma} \mathbf{U}(g - \chi(g))\big). \tag{4.1}$$

This follows from the triangular decomposition in Corollary 3.8 (2).

Let $m_\chi \in M(\chi)$ be the residue class of 1 in $M(\chi)$. Then $M(\chi) \in \mathcal{C}$, $m_\chi \in M(\chi)^\chi$, and $E_i m_\chi = 0$ for all $1 \le i \le \theta$. The pair $(M(\chi), m_\chi)$ has the following universal property: For any $M \in \mathcal{C}$ with $m \in M^\chi$ such that $E_i m = 0$ for all $1 \le i \le \theta$ there exists a unique $\mathbf{U}$-linear map $t : M(\chi) \to M$ such that $t(m_\chi) = m$.

The Verma module $M(\chi)$ and all its quotients belong to the category $\mathcal{C}^{hi}$. We define a partial order $\le$ on $\widehat{\Gamma}$.

Definition 4.2. For all $\chi, \chi' \in \widehat{\Gamma}$ we write $\chi' \le \chi$ if there is an element $\alpha \in \mathbb{N}^{\mathbb{I}}$ such that $\chi = \chi'\chi_\alpha$.

Note that $\le$ is a partial order in $\widehat{\Gamma}$ since $\mathcal{D}_{red}$ is regular.

Lemma 4.3. *Let $\chi \in \widehat{\Gamma}$, and $M \in \mathcal{C}$. Suppose χ is a maximal weight for M and $m \in M^\chi$. Then $E_i m = 0$ for all $1 \le i \le \theta$, and $\mathbf{U}m$ is a quotient of $M(\chi)$.*

Proof. This follows from the universal property of the Verma module since $E_i m \in M^{\chi\chi_{\alpha_i}}$, and $M^{\chi\chi_{\alpha_i}} = 0$ by maximality of χ. □

Let $M \in \mathcal{C}$ and C be a coset of Q in $\widehat{\Gamma}$. Then $M_C = \oplus_{\chi \in C} M^\chi$ is an object of $\mathcal{C}$. We note that $M = \oplus_C M_C$, where C runs over the Q-cosets of $\widehat{\Gamma}$.

By regularity, for all $\chi \in \widehat{\Gamma}$

$$M(\chi) = \oplus_{\alpha\in\mathbb{N}^{\mathbb{I}}} M(\chi)^{\chi(\chi_\alpha)^{-1}},\ M(\chi)^\chi = \Bbbk m_\chi,$$

since $M(\chi)$ is the $\Bbbk$-span of the residue classes of $F_{i_1} \cdots F_{i_n}, 1 \le i_1, \cdots, i_n \le \theta, n \ge 0$. Thus χ is a weight of $M(\chi)$ with one-dimensional weight space, $\chi' \le \chi$ for all weights χ' of $M(\chi)$, and $M(\chi) = (M(\chi))_C$, where $C = \chi Q$. Because of these remarks, the proof of the following Lemma is standard.

Lemma 4.4. *If $\chi \in \widehat{\Gamma}$, then $M(\chi)$ has a unique maximal submodule $M'(\chi)$; the quotient $\Lambda(\chi) := M(\chi)/M'(\chi)$ is the unique (up to isomorphisms) simple module with highest weight χ.* □

Lemma 4.5. *Suppose $M \in \mathcal{C}^{hi}$ is finitely generated as a $\mathbf{U}$-module.*

1. *The dimension of M^χ is finite for all $\chi \in \widehat{\Gamma}$.*
2. *For all $\chi' \in \widehat{\Gamma}$ there are only finitely many weights χ for M which satisfy $\chi' \le \chi$.*
3. *Every non-empty set of weights for M has a maximal element.*

Proof. We may assume that $M \neq 0$. In this case M is generated by weight vectors $v_1, \dots, v_r$. Let $\chi_1, \dots, \chi_r \in \widehat{\Gamma}$ be the corresponding weights. Let χ be a weight for M. Observe that M^χ is spanned by elements of the form

$$0 \neq m = F_{i_1} \cdots F_{i_s} E_{j_1} \cdots E_{j_t} g \cdot v_i = \chi_i(g) F_{i_1} \cdots F_{i_s} E_{j_1} \cdots E_{j_t} \cdot v_i,$$

where $1 \leq i \leq r$, $g \in \Gamma$, $0 \leq s, t$, $1 \leq i_p, j_q \leq \theta$ for all $1 \leq p \leq s$, $1 \leq q \leq t$, and $\chi = \chi_{-\beta+\alpha}\chi_i$, where $\beta = \alpha_{i_1} + \cdots + \alpha_{i_s}$, and $\alpha = \alpha_{j_1} + \cdots + \alpha_{j_t}$. Since $M \in \mathcal{C}^{hi}$ there are only finitely many α's for each $1 \leq i \leq r$, and for each pair (α, i) there is exactly one β with $\chi = \chi_{-\beta+\alpha}\chi_i$. Here we use the fact that $\chi_1, \dots, \chi_\theta$ are $\mathbb{Z}$-linearly independent, that is for $\alpha, \beta \in \mathbb{Z}^{\mathbb{I}}$ the equations $\chi_\alpha = \chi_\beta$ implies $\alpha = \beta$. We have established (1).

Let $\chi' \in \widehat{\Gamma}$ and χ a weight for M such that $\chi' \leq \chi$. Note that $\chi \leq \chi_\alpha \chi_i$ for some α and i as above. This proves (2) since there are only finitely many such pairs (α, i), and since for all $\chi_1, \chi_2 \in \widehat{\Gamma}$ the segment

$$[\chi_1, \chi_2] = \{\varphi \in \widehat{\Gamma} \mid \chi_1 \leq \varphi \leq \chi_2\}$$

is finite. A consequence of (2) is that every chain of weights $\chi_1 \leq \chi_2 \leq \cdots$ is finite; hence (3) follows. □

4.2. **Integrable modules.**

A left $\mathbf{U}$-module M is called *integrable* if $M \in \mathcal{C}$, and for any $m \in M$ and $1 \leq i \leq \theta$ there is a natural number $n \geq 1$ such that $E_i^n m = F_i^n m = 0$.

The following notion from [RS2] is an adaptation to the present setting of the classical concept in Lie theory. A character $\chi \in \widehat{\Gamma}$ is called *dominant* if there are natural numbers $m_i \geq 0$ such that $\chi(K_i L_i) = q_{ii}^{m_i}$ for all $1 \leq i \leq \theta$.

We denote the set of all dominant characters in $\widehat{\Gamma}$ by $\widehat{\Gamma}^+$.

Definition 4.6. Let $\chi \in \widehat{\Gamma}^+$ and $m_i \geq 0$ for all $1 \leq i \leq \theta$ such that $\chi(K_i L_i) = q_{ii}^{m_i}$ for all $1 \leq i \leq \theta$. Set

$$L_{\mathbf{U}}(\chi) = \mathbf{U}/\big(\sum_{i=1}^{\theta} \mathbf{U} E_i + \sum_{i=1}^{\theta} \mathbf{U} F_i^{m_i+1} + \sum_{g \in \Gamma} \mathbf{U}(g - \chi(g))\big).$$

We will write $L(\chi) = L_{\mathbf{U}}(\chi)$ when the Hopf algebra $\mathbf{U}$ is fixed.

Lemma 4.7. *Let $n \geq 1$ and $1 \leq i \leq \theta$. Then*

(1) $E_i F_i^n = F_i^n E_i + \ell_i \dfrac{q_{ii}^n - 1}{q_{ii} - 1}(K_i - L_i^{-1} q_{ii}^{-n+1}) F_i^{n-1}$.

(2) $F_i E_i^n = E_i^n F_i + \ell_i \dfrac{q_{ii}^n - 1}{q_{ii} - 1}(L_i^{-1} - K_i q_{ii}^{-n+1}) E_i^{n-1}$.

(3) $F_i^n F_j \in \sum_{s=0}^{-aij} \Bbbk F_i^s F_j F_i^{n-s}$, *if* $n \geq 1 - a_{ij}$.

(4) $E_i^n E_j \in \sum_{s=0}^{-aij} \Bbbk E_i^s E_j E_i^{n-s}$, *if* $n \geq 1 - a_{ij}$.

Proof. (1) and (2) follow from Prop. 3.13 (1) and Prop. 3.12 (1), or can be shown directly by induction on n. (3) and (4) follow from the Serre relations (3.39) and (3.38) and the observation that for a, b in an algebra and $r \geq 1$ the relation $a^r b \in \sum_{s=0}^{r-1} \Bbbk a^s b a^{r-s}$ implies $a^n b \in \sum_{s=0}^{r-1} \Bbbk a^s b a^{n-s}$ for all $n \geq r$. □

Let χ be dominant, and let ℓ_χ be the residue class of 1 in $L(\chi)$. By the next lemma the pair $(L(\chi), \ell_\chi)$ has the universal property of the Verma module with respect to integrable modules in $\mathcal{C}$.

Proposition 4.8. *Let $M \in \mathcal{C}$ be integrable and $\chi \in \widehat{\Gamma}$. Assume that there exists an element $0 \neq m \in M^\chi$ such that $E_i m = 0$ for all $1 \leq i \leq \theta$. Then χ is dominant, and there is a unique $\mathbf{U}$-linear map $t : L(\chi) \to M$ such that $t(\ell_\chi) = m$.*

Proof. Let $1 \leq i \leq \theta$. Since M is integrable, there is an integer $n \geq 1$ such that $F_i^n m = 0$, $F_i^{n-1} m \neq 0$. By Lemma 4.7 (1)

$$\begin{aligned}
0 = E_i F_i^n m &= \ell_i \frac{q_{ii}^n - 1}{q_{ii} - 1}(K_i - L_i^{-1} q_{ii}^{-n+1}) F_i^{n-1} m \\
&= \ell_i \frac{q_{ii}^n - 1}{q_{ii} - 1}\left(\chi(K_i) q_{ii}^{-n+1} - \chi(L_i^{-1})\right) F_i^{n-1} m,
\end{aligned}$$

since $F_i^{n-1} m \in M^{\chi \chi_i^{1-n}}$. Since q_{ii} is not a root of unity, it follows that $\chi(K_i L_i) = q_{ii}^{n-1}$. Hence $n = m_i + 1$. Thus χ is dominant, and the universal $\mathbf{U}$-linear map $M(\chi) \to M$, $m_\chi \mapsto m$, factorizes over $L(\chi)$. □

Corollary 4.9. *Let $\chi, \chi' \in \widehat{\Gamma}^+$.*

(1) *The isomorphism* (4.1) *induces an isomorphism*

$$\mathbf{U}^- / \big(\sum_{i=1}^{\theta} \mathbf{U}^- F_i^{m_i+1} \big) \xrightarrow{\cong} L(\chi), \tag{4.2}$$

$L(\chi)$ is integrable, and $\dim L(\chi)^\chi = 1$, *with basis ℓ_χ.*

(2) *The modules $L(\chi)$ and $L(\chi')$ are isomorphic if and only if $\chi = \chi'$.*

Proof. (1) By Lemma 4.8 $E_i F_i^{m_i+1} \overline{1} = 0$ in $M(\chi)$, $1 \leq i \leq \theta$. Hence the image of $\mathbf{U} F_i^{m_i+1}$ in $M(\chi)$ coincides with the image of $\mathbf{U}^- F_i^{m_i+1}$, and the map in (1) is bijective. In particular, $L(\chi) \neq 0$, and $L(\chi)^\chi$ is one-dimensional with basis $\overline{1} = l_\chi$. By Lemma 4.7 (3) $L(\chi)$ is integrable. (2) follows from (1) since $L(\chi)$ and $L(\chi')$ have unique highest weights χ and χ'. □

We note that in the proof of the last corollary we used the following rule in $\mathbf{U}^-$ to show that $L(\chi)$ is integrable: For all $1 \leq i, j \leq \theta, i \neq j$, there are integers $n_{ij} \geq r_{ij} \geq 0$ such that

$$F_i^n F_j \in \mathbf{U}^- F_i^{n - r_{ij}} \quad \text{for all } n \geq n_{ij}. \tag{4.3}$$

This rule follows from Lemma 4.7 (3) with $r_{ij} = -a_{ij}, n_{ij} = 1 - a_{ij}$, that is, from the Serre relations which hold because $\mathcal{D}_{red}$ is of Cartan type. The

assumption of Cartan type is only used here. Thus in Section 4 we could replace it by (4.3).

4.3. **The quantum Casimir operator.**

We assume in this subsection the following condition on the diagonal entries of the braiding matrix (q_{ij}):

$$\text{(4.4)} \quad \text{If } \prod_{i=1}^{\theta} q_{ii}^{n_i} = 1, 0 \le n_i \in \mathbb{Z}, 1 \le i \le \theta, \text{ then } n_i = 0 \text{ for all } 1 \le i \le \theta.$$

By Lemma 3.20 (4.4) holds if $\mathbb{I}$ is connected, that is if the Cartan matrix of $\mathcal{D}_{red}$ is indecomposable. As in [L2, Chapter 6] the next lemma is crucial for the semisimplicity results.

Lemma 4.10. *Let C be a coset of Q in $\widehat{\Gamma}$.*

1. *There is a function $G : C \to \Bbbk^\times$ such that $G(\chi) = G(\chi\chi_i^{-1})\chi(K_iL_i)$, for all $\chi \in C$ and $1 \le i \le \theta$. G is uniquely determined up to multiplication by a constant in $\Bbbk^\times$.*
2. *Let G be as in (1). If $\chi, \chi' \in \widehat{\Gamma}^+$ are dominant characters with $\chi \ge \chi'$ and $G(\chi) = G(\chi')$, then $\chi = \chi'$.*

Proof. (1) Let $C = \bar{\chi}Q$ where $\bar{\chi}$ is a fixed element in the coset C, and pick $G(\bar{\chi}) \in \Bbbk^\times$. For all $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{Z}^{\mathbb{I}}$ we define

$$q_\alpha = \prod_{i=1}^{\theta} q_{ii}^{n_i(n_i+1)} \prod_{1\le i<j\le\theta} (q_{ij}q_{ji})^{n_in_j}, \tag{4.5}$$

$$G(\bar{\chi}\chi_\alpha) = G(\bar{\chi})\bar{\chi}(K_\alpha L_\alpha)q_\alpha. \tag{4.6}$$

We first show that for all $\alpha \in \mathbb{Z}^{\mathbb{I}}, 1 \le p \le \theta$,

$$q_\alpha = q_{\alpha-\alpha_p}\chi_\alpha(K_pL_p). \tag{4.7}$$

By definition

$$\begin{aligned} q_{\alpha-\alpha_p} &= \prod_{\substack{1\le i\le\theta,\\ i\ne p}} q_{ii}^{n_i(n_i+1)} q_{pp}^{(n_p-1)n_p} \prod_{\substack{1\le i<j\le\theta,\\ i\ne p, j\ne p}} (q_{ij}q_{ji})^{n_in_j} \prod_{\substack{1\le i\le\theta,\\ i\ne p}} (q_{ip}q_{pi})^{n_pn_i-n_i} \\ &= \prod_{1\le i\le\theta} q_{ii}^{n_i(n_i+1)} q_{pp}^{-2n_p} \prod_{1\le i<j\le\theta} (q_{ij}q_{ji})^{n_in_j} \prod_{\substack{1\le i\le\theta,\\ i\ne p}} (q_{ip}q_{pi})^{-n_i} \\ &= q_\alpha\chi_\alpha(K_pL_p)^{-1}, \end{aligned}$$

where the last equality follows from $\chi_i(K_pL_p) = q_{ip}q_{pi}$ for all $1 \le i \le \theta$.

It follows from (4.7) that the function G defined by (4.6) has the desired property since for all $\alpha \in \mathbb{Z}^{\mathbb{I}}, 1 \leq p \leq \theta$,

$$\begin{aligned}
G(\bar{\chi}\chi_\alpha\chi_p^{-1})(\bar{\chi}\chi_\alpha)(K_pL_p) &= G(\bar{\chi})\bar{\chi}(K_{\alpha-\alpha_p}L_{\alpha-\alpha_p})q_{\alpha-\alpha_p}\bar{\chi}(K_pL_p)\chi_\alpha(K_pL_p) \\
&= G(\bar{\chi})\bar{\chi}(K_\alpha L_\alpha)\bar{\chi}(K_p^{-1}L_p^{-1})q_{\alpha-\alpha_p}\bar{\chi}(K_pL_p)\chi_\alpha(K_pL_p) \\
&= G(\bar{\chi})\bar{\chi}(K_\alpha L_\alpha)q_{\alpha-\alpha_p}\chi_\alpha(K_pL_p) \\
&= G(\bar{\chi})\bar{\chi}(K_\alpha L_\alpha)q_\alpha \\
&= G(\bar{\chi}\chi_\alpha).
\end{aligned}$$

The functions G in (1) are clearly unique up to a non-zero scalar.

(2) (a) We show by induction on $n \geq 0$ that for all $1 \leq i_1, i_2, \ldots, i_n \leq \theta$,

$$\frac{G(\chi)}{G(\chi\chi_{i_1}^{-1}\chi_{i_2}^{-1}\cdots\chi_{i_n}^{-1})} = \prod_{1\leq p\leq n}\chi(K_{i_p}L_{i_p})\prod_{1\leq p<q\leq n}\chi_{i_p}(K_{i_q}L_{i_q})^{-1}. \tag{4.8}$$

This is clear for $n = 0$ and follows by induction and (1) from

$$\begin{aligned}
&G(\chi)G(\chi\chi_{i_1}^{-1}\cdots\chi_{i_{n+1}}^{-1})^{-1} \\
&= G(\chi)G(\chi\chi_{i_1}^{-1}\cdots\chi_{i_n}^{-1})^{-1}G(\chi\chi_{i_1}^{-1}\cdots\chi_{i_n}^{-1})G(\chi\chi_{i_1}^{-1}\cdots\chi_{i_{n+1}}^{-1})^{-1} \\
&= \prod_{1\leq p\leq n}\chi(K_{i_p}L_{i_p})\prod_{1\leq p<q\leq n}\chi_{i_p}(K_{i_q}L_{i_q})^{-1}(\chi\chi_{i_1}^{-1}\cdots\chi_{i_n}^{-1})(K_{i_{n+1}}L_{i_{n+1}}) \\
&= \prod_{1\leq p\leq n+1}\chi(K_{i_p}L_{i_p})\prod_{1\leq p<q\leq n+1}\chi_{i_p}(K_{i_q}L_{i_q})^{-1}.
\end{aligned}$$

(b) Now we prove (2). By assumption there are indices $1 \leq i_1, i_2, \ldots, i_n \leq \theta$, $n \geq 0$, with $\chi' = \chi\chi_{i_1}^{-1}\cdots\chi_{i_n}^{-1}$. Since χ and χ' are both dominant there are natural numbers $m_i, m_i' \geq 0$, $1 \leq i \leq \theta$, such that

$$\chi(K_iL_i) = q_{ii}^{m_i},\ \chi'(K_iL_i) = q_{ii}^{m_i'} \text{ for all } 1 \leq i \leq \theta. \tag{4.9}$$

By assumption $G(\chi) = G(\chi') = G(\chi\chi_{i_1}^{-1}\cdots\chi_{i_n}^{-1})$. Hence by (a),

$$\prod_{1\leq p\leq n}\chi(K_{i_p}L_{i_p}) = \prod_{1\leq p<q\leq\theta}\chi_{i_p}(K_{i_q}L_{i_q}). \tag{4.10}$$

Then we obtain

$$\begin{aligned}
\prod_{1\leq p\leq n}q_{i_pi_p}^{m_{i_p}+m'_{i_p}} &= \prod_{1\leq p\leq n}\chi(K_{i_p}L_{i_p})\prod_{1\leq p\leq n}\chi'(K_{i_p}L_{i_p}) \\
&= \prod_{1\leq p\leq n}\chi(K_{i_p}L_{i_p})^2\prod_{1\leq p,q\leq n}\chi_{i_q}^{-1}(K_{i_p}L_{i_p}) \\
= \prod_{1\leq p<q\leq n}\chi_{i_p}(K_{i_q}L_{i_q})^2 &\prod_{1\leq p<q\leq n}\chi_{i_p}(K_{i_q}L_{i_q})^{-2}\times\prod_{1\leq p\leq n}\chi_{i_p}^{-1}(K_{i_p}L_{i_p}) \\
&= \prod_{1\leq p\leq n}q_{i_pi_p}^{-2}.
\end{aligned}$$

For the third equality we used (4.10) and that $\chi_i(K_jL_j) = q_{ij}q_{ji} = \chi_j(K_iL_i)$ for all $1 \le i, j \le \theta$. By (4.4) the family $(q_{ii})_{1\le i\le\theta}$ is $\mathbb{N}$-linearly independent and we get a contradiction except $n = 0$, that is $\chi = \chi'$. $\square$

Example 4.11. Let $\Gamma = \langle K_1, K_2\rangle$ be a free abelian group with basis K_1, K_2, and $0 \ne q \in \Bbbk$ not a root of unity. Let $L_1 = K_1$, $L_2 = K_2$, and define characters $\chi_1, \chi_2 \in \widehat{\Gamma}$ by

$$\chi_1(K_1) = q, \quad \chi_1(K_2) = 1, \quad \chi_2(K_1) = 1, \quad \chi_2(K_2) = q^{-1}.$$

Thus $\mathcal{D}_{red} = \mathcal{D}_{red}(\Gamma, (L_i), (K_i), (\chi_i), (a_{ij}))$ is a generic reduced YD-datum of Cartan type where $a_{11} = a_{22} = 2, a_{12} = a_{21} = 0$, and $q_{11}q_{22} = 1$. Define $\chi, \chi' \in \widehat{\Gamma}$ by $\chi'(K_1) = q$, $\chi'(K_2) = q^{-1}$, and $\chi = \chi'\chi_1\chi_2$. Then $\chi' \le \chi$, and both are dominant. Let G be a function satisfying Lemma 4.10 (1) for the coset $C = \chi' Q$. Then

$$G(\chi) = G(\chi'\chi_1\chi_2) = G(\chi')(\chi'\chi_1)(K_1L_1)(\chi'\chi_1\chi_2)(K_2L_2) = G(\chi').$$

Thus Lemma 4.10 (2) does not hold without the assumption that the q_{ii}'s are $\mathbb{N}$-linearly independent.

Proposition 4.12. *Let C be a coset of Q in $\widehat{\Gamma}$, and $M \in \mathcal{C}^{hi}$ such that $M = M_C$. Choose a function G as in Lemma 4.10 and define a $\Bbbk$-linear map $\Omega_G : M \to M$ by $\Omega_G(m) = G(\chi)\Omega(m)$ for all $m \in M^\chi, \chi \in C$.*

1. *The map Ω_G is $\mathbf{U}$-linear and locally finite.*
2. *If $0 \ne m \in M$ generates a quotient of a Verma module $M(\chi)$ for some $\chi \in \widehat{\Gamma}$, then $\chi \in C$, and $\Omega_G(m) = G(\chi)m$.*
3. *The eigenvalues of Ω_G are the $G(\chi)'s$, where χ runs over the maximal weights of the submodules N of M, in which case $\Omega_G(n) = G(\chi)n$ for all $n \in N^\chi$.*

Proof. (1) By Proposition 4.1 Ω_G is well-defined and maps each weight space of M to itself. Hence $\Omega_G : M \to M$ is Γ-linear. Let $1 \le i \le \theta, \chi \in \widehat{\Gamma}$, and $m \in M^\chi$. By Propositon 4.1

$$\Omega_G(E_im) = G(\chi_i\chi)\Omega(E_im) = G(\chi_i\chi)(\chi\chi_i)^{-1}(K_iL_i)E_i\Omega m,$$
$$\Omega_G(F_im) = G(\chi_i^{-1}\chi)\Omega(F_im) = G(\chi_i^{-1}\chi)\chi(K_iL_i)F_i\Omega m.$$

On the other hand, $E_i\Omega_G(m) = G(\chi)E_i\Omega m, F_i\Omega_G(m) = G(\chi)F_i\Omega m$. By Lemma 4.10 (1),

$$G(\chi) = G(\chi\chi_i^{-1})\chi(K_iL_i), G(\chi\chi_i) = G(\chi)(\chi\chi_i)(K_iL_i).$$

Hence it follows that

$$\Omega_G(E_im) = E_i\Omega_G(m),\ \Omega_G(F_im) = F_i\Omega_G(m),$$

and we have shown that Ω_G is $\mathbf{U}$-linear. We show that M is the sum of finite-dimensional Ω_G-invariant subspaces. Since any $\mathbf{U}$-submodule of M is Ω_G-invariant we may assume that M is finite-dimensional. In this case M is

the sum of finite-dimensional weight spaces by Lemma 4.5 (1), and weight spaces are Ω_G-invariant. Hence Ω_G is a locally finite linear map.

(2) Write $\mathbf{U}\cdot m = \mathbf{U}\cdot n$, where $n \in M^\chi$ and $E_i\cdot n = 0$ for all $1 \le i \le \theta$. Then $\Omega_G(n) = G(\chi)n$ by definition of Ω_G and consequently $\Omega_G(n') = G(\chi)n'$ for all $n' \in \mathbf{U}\cdot n$ since eigenspaces of module endomorphisms are submodules.

(3) First of all, $G(\chi)$ is an eigenvalue of Ω_G when χ is a maximal weight of submodule of M by Lemma 5.3 and part (2).

Conversely, suppose that λ is an eigenvalue of Ω_G and $0 \neq m \in M$ satisfies $\Omega_G(m) = \lambda m$. Since $N = U \cdot m \neq 0$ is finitely generated, and $N \in \mathcal{C}^{hi}$, by Lemma 5.4 (3) there is a maximal weight χ for N. By Lemma 5.3 and part (2) we conclude that $G(\chi)$ is an eigenvalue for the restriction $\Omega_G|N$. Since the eigenvectors for Ω_G belonging to λ form a submodule of M, $G(\chi) = \lambda$. □

The function $\Omega_G : M \to M$ in Proposition 4.12 is called the *quantum Casimir operator*.

4.4. Irreducible highest weight modules.

Lemma 4.13. *Let $\chi \in \widehat{\Gamma}^+$. Let J be a connected component of $\mathbb{I}$, $\mathbb{I}' = \mathbb{I} \setminus J$, and let $\mathbf{U}_J$ be the subalgebra of $\mathbf{U}$ generated by Γ and $E_j, F_j, j \in J$ and $\mathbf{U}'$ the subalgebra of $\mathbf{U}$ generated by Γ and $E_i, F_i, i \in \mathbb{I}'$. Then the map*

$$\Phi : L_{\mathbf{U}_J}(\chi) \otimes L_{\mathbf{U}'}(\chi) \to L(\chi), \quad \overline{u} \otimes \overline{u'} \mapsto \overline{uu'}, \tag{4.11}$$

for all $u \in \mathbf{U}_J^-$, $u' \in \mathbf{U}'^-$, is a $\Bbbk$-linear isomorphism; and

1. *$\Phi(gm \otimes gm') = \chi(g)\, g\Phi(m \otimes m')$,*
2. *$\Phi(E_j m \otimes m') = (\chi\psi^{-1})(K_j)\, E_j \Phi(m \otimes m')$, if $m' \in L_{\mathbf{U}'}(\chi)^\psi$, $\psi \in \widehat{\Gamma}$,*
3. *$\Phi(m \otimes E_i m') = E_i \Phi(m \otimes m')$,*

for all $j \in J$, $i \in \mathbb{I}'$, $m \in L_{\mathbf{U}_J}(\chi)$, $m' \in L_{\mathbf{U}'}(\chi)$.

Proof. The multiplication map defines an isomorphism $\mathbf{U}_J^- \otimes \mathbf{U}'^- \to \mathbf{U}^-$ since the generators $F_j, j \in J$, of $\mathbf{U}_J^-$ and $F_i, i \in \mathbb{I}'$, of $\mathbf{U}'^-$ skew-commute. The kernel of the canonical map

$$\mathbf{U}_J^- \otimes \mathbf{U}' \to \mathbf{U}_J^-/\big(\sum_{j\in J} \mathbf{U}_J^- F_j^{m_j+1}\big) \otimes \mathbf{U}'^-/\big(\sum_{i\in\mathbb{I}'} \mathbf{U}'^- F_i^{m_i+1}\big)$$

has image

$$\sum_{j\in J} \mathbf{U}_J^- F_j^{m_j+1}\mathbf{U}' + \mathbf{U}_J^- \sum_{i\in\mathbb{I}'} \mathbf{U}'^- F_i^{m_i+1} = \sum_{i\in\mathbb{I}} \mathbf{U} F_i^{m_i+1}$$

under the multiplication map. Hence the induced map

$$\mathbf{U}_J^-/\big(\sum_{j\in J} \mathbf{U}_J^- F_j^{m_j+1}\big) \otimes \mathbf{U}'^-/\big(\sum_{i\in\mathbb{I}'} \mathbf{U}'^- F_i^{m_i+1}\big) \to \mathbf{U}^-/\big(\sum_{i\in\mathbb{I}} \mathbf{U}^- F_i^{m_i+1}\big)$$

is bijective. Then (4.11) is an isomorphism of vector spaces, by Corollary 4.9.

To prove (1) – (3), we may assume that $m = u\,\overline{1}$ and $m' = u'\,\overline{1}$, where $u \in (\mathbf{U}_J^-)_{-\alpha}$, $u' \in (\mathbf{U}'^-)_{-\beta}$ are homogeneous with $\alpha \in \mathbb{N}^J$, $\beta \in \mathbb{N}^{\mathbb{I}'}$.

(1) Let $g \in \Gamma$. Then

$$\begin{aligned}\Phi(gu\overline{1} \otimes gu'\overline{1})) &= (\chi_{-\alpha}\chi)(g)(\chi_{-\beta}\chi)(g)\Phi(u\overline{1} \otimes u'\overline{1})\\ &= (\chi_{-\alpha}\chi)(g)(\chi_{-\beta}\chi)(g)uu'\overline{1}\\ &= \chi(g)guu'\overline{1}\\ &= \chi(g)g\Phi(u\overline{1} \otimes u'\overline{1}).\end{aligned}$$

(2) Let $j \in J$. We first note that there is an element $\widetilde{u} \in \mathbf{U}_J$ which is a $\Bbbk$-linear combination of monomials in $F_l, l \in J$ and $K_j - L_j^{-1}$ where in each monomial the factor $K_j - L_j^{-1}$ occurs exactly one, and such that $E_j u = uE_j + \widetilde{u}$. This follows by induction on $|\alpha|$, since

$$E_jF_ku = F_kE_ju + \delta_{jk}l(K_j - L_j^{-1})u = F_kuE_j + F_k\widetilde{u} + \delta_{jk}l(K_j - L_j^{-1})u$$

for all $k \in J$ by induction and (3.7). Then

$$\begin{aligned}&\Phi(E_ju\overline{1} \otimes u'\overline{1}) = \Phi(\widetilde{u}\overline{1} \otimes u'\overline{1}),\\ &E_j\Phi(u\overline{1} \otimes u'\overline{1}) = E_juu'\overline{1} = (uE_j + \widetilde{u})u'\overline{1} = \widetilde{u}u'\overline{1},\end{aligned}$$

since $E_j\overline{1} = 0$ in $L_{\mathbf{U}_J}(\chi)$, and $E_ju'\overline{1} = u'E_j\overline{1} = 0$ in $L_{\mathbf{U}'}(\chi)$. Hence (2) is equivalent to

$$\Phi(\widetilde{u}\overline{1} \otimes u'\overline{1}) = \chi_\beta(K_j)\widetilde{u}u'\overline{1}, \tag{4.12}$$

since $u'\overline{1} \in L_{\mathbf{U}'}(\chi)^{\chi\chi_{-\beta}}$, hence $\chi\psi^{-1} = \chi_\beta$. To prove (4.12) we may assume that $\widetilde{u} = u_1(K_j - L_j^{-1})u_2$, where $u_1 \in \mathbf{U}_j^-$ and $u_2 \in (\mathbf{U}_J^-)_{-\gamma}, \gamma \in \mathbb{N}^J$. Then

$$\widetilde{u}\overline{1} = u_1(K_j - L_j^{-1})u_2\overline{1} = ((\chi\chi_{-\gamma})(K_j) - (\chi\chi_{-\gamma})(L_j)^{-1})u_1u_2\overline{1}$$

in $L_J(\chi)$, and $\Phi(\widetilde{u}\overline{1} \otimes u'\overline{1}) = ((\chi\chi_{-\gamma})(K_j) - (\chi\chi_{-\gamma})(L_j)^{-1})u_1u_2u'\overline{1}$.

Since $\chi_i(K_j) = q_{ji} = q_{ij}^{-1}$, and $\chi_i(L_j)^{-1} = \chi_j(K_i)^{-1} = q_{ij}^{-1}$ for all $i \in \mathbb{I}'$ it follows that in $L(\chi)$

$$\begin{aligned}\widetilde{u}u'\overline{1} &= u_1(K_j - L_j^{-1})u_2u'\overline{1}\\ &= ((\chi\chi_{-\gamma}\chi_{-\beta})(K_j) - (\chi\chi_{-\gamma}\chi_{-\beta})(L_j)^{-1})u_1u_2u'\overline{1}\\ &= \chi_{-\beta}(K_j)((\chi\chi_{-\gamma})(K_j) - (\chi\chi_{-\gamma})(L_j)^{-1})u_1u_2u'\overline{1}\\ &= \chi_{-\beta}(K_j)\Phi(\widetilde{u}\overline{1} \otimes u'\overline{1}).\end{aligned}$$

(3) As in the proof of (2) let $\widetilde{u'} \in \mathbf{U}'$ with $E_iu = uE_i + \widetilde{u'}$. Then

$$\begin{aligned}&\Phi(u\overline{1} \otimes E_iu'\overline{1}) = \Phi(u\overline{1} \otimes \widetilde{u'}\overline{1}),\\ &E_i\Phi(u\overline{1} \otimes u'\overline{1}) = E_iuu'\overline{1} = uE_iu'\overline{1} = u\widetilde{u'}\overline{1}.\end{aligned}$$

To prove that $\Phi(u\overline{1} \otimes \widetilde{u'}) = u\widetilde{u'}\overline{1}$ we may assume that

$$\widetilde{u'} = u_1'(K_i - L_i^{-1})u_2', u_2' \in (\mathbf{U}'^-)_{-\delta}, \delta \in \mathbb{N}^{\mathbb{I}'}.$$

Then $\Phi(u\overline{1} \otimes \widetilde{u'}\overline{1}) = ((\chi\chi_{-\delta})(K_i) - (\chi\chi_{-\delta})(L_i)^{-1})uu_1u_2\overline{1} = u\widetilde{u'}\overline{1}$. □

The following theorem shows that $L(\chi)$ for $\chi \in \widehat{\Gamma}^+$ coincides with the simple module $\Lambda(\chi)$ from Lemma 4.4, thus providing the defining relations of $\Lambda(\chi)$; it also implies that the weight multiplicities of $\Lambda(\chi)$ are as in the classical case for data of finite Cartan type.

Remark 4.14. Let $(a_{ij})_{1\le i,j\le\theta}$ be a symmetrizable Cartan matrix, $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of $(a_{ij})_{1\le i,j\le\theta}$ with $\Pi = \{\alpha_1, \dots, \alpha_\theta\}, \Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\theta^\vee\}$ and $\mathfrak{g}$ the corresponding Kac-Moody Lie algebra (see [K]). let $0 \ne q \in \Bbbk$ be not a root of unity and (V, c) a braided vector space with basis $v_1, \dots, v_\theta$ and braiding $c(v_i \otimes v_j) = q^{d_i a_{ij}} v_j \otimes v_i$ for all $1 \le i, j \le \theta$, where $(d_i a_{ij})$ is the symmetrized Cartan matrix.

Let $0 \le m_1, \dots, m_\theta \in \mathbb{Z}$. Choose $\lambda \in \mathfrak{h}^*$ with $\lambda(\alpha_i^\vee) = m_i$ for all $1 \le i \le \theta$. Thus λ is an integral weight of $\mathfrak{g}$. Let $L(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight λ. Then the multiplicities of the weight spaces of $L(\lambda)$ are given by

$$L(\lambda)_{\lambda-\alpha} \cong (U(\mathfrak{n}_+)/(\sum_{i=1}^{\theta} U(\mathfrak{n}_+)e_i^{m_i+1})_\alpha \cong (\mathcal{B}(V)/(\sum_{i=1}^{\theta} \mathcal{B}(V)v_i^{m_i+1}))_\alpha,$$

where $\deg(e_i) = \deg(v_i) = \alpha_i$ for all i, and $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i, 0 \le n_i \in \mathbb{Z}$ for all i. The first isomorphism is [K, 10.4.6], and the second isomorphism follows from [L2, 33.1.3] if q is transcendental, and can be derived from [HK, Section 3.4] if q is not a root of unity.

Theorem 4.15. *Let* $\chi \in \widehat{\Gamma}^+$.

1. $L(\chi)$ *is a simple* $\mathbf{U}$*-module.*
2. *Any weight vector of* $L(\chi)$ *which is annihilated by all* $E_i, 1 \le i \le \theta$, *is a scalar multiple of* ℓ_χ.
3. *If* (q_{ij}) *satisfies* (3.40)*, in particular if the Cartan matrix is of finite type, then the weight multiplicities are as in the classical case, that is, given by the Weyl-Kac character formula.*

Proof. We proceed by induction on the number of connected components of $\mathbb{I}$. We first assume that $\mathbb{I}$ is connected. Then the results of Subsection 4.3 apply. Recall that $L(\chi) = L(\chi)_C$ for the coset $C = \chi Q$.

(1) Let M be a non-zero submodule of $L(\chi)$. By Lemma 4.5 (3) there is a maximal weight for M since $L(\chi)$ is finitely generated. Let χ' be such a weight. Then $G(\chi')$ is an eigenvalue for Ω_G by Prop. 4.12 (3). By part (2) of the same $G(\chi) = G(\chi')$. Since $L(\chi)$ is integrable M is also. By Lemma 4.3 and Prop. 4.8 χ' is dominant. Thus $\chi = \chi'$ by Lemma 4.10 (2). Since $L(\chi)^\chi$ is one-dimensional $L(\chi)^\chi = M^\chi$. Thus $M = L(\chi)$ since $L(\chi)^\chi$ generates $L(\chi)$. Thus we have shown that $L(\chi)$ is simple.

(2) Let $\chi' \in \widehat{\Gamma}$ and $0 \ne m \in L(\chi)^{\chi'}$ such that $E_i m = 0$ for all $1 \le i \le \theta$. By Proposition 4.8 χ' is dominant and there is a $\mathbf{U}$-linear map $L(\chi') \to L(\chi)$

mapping $\ell_{\chi'}$ onto m. This map is an isomorphism since $L(\chi')$ and $L(\chi)$ are simple by the first part of the proof. Hence $\chi' = \chi$ by Corollary 4.9 (2), and m is a scalar multiple of ℓ_χ by Corollary 4.9 (1).

(3) Let $\chi \in \widehat{\Gamma}^+$ and $\chi(K_i L_i) = q_{ii}^{m_i}, 0 \leq m_i \in \mathbb{Z}$, for all $1 \leq i \leq \theta$. By Corollary 3.9 the weights of $L(\chi)$ have the form $\chi\chi_{-\alpha}, \alpha \in \mathbb{N}^{\mathbb{I}}$, where for all $\alpha \in \mathbb{N}^{\mathbb{I}}$, $L(\chi)^{\chi\chi_{-\alpha}} \cong \big(\mathbf{U}^-/\big(\sum_{i=1}^{\theta} \mathbf{U}^- F_i^{m_i+1}\big)\big)_{-\alpha}$. The bijective map $\widetilde{\kappa} : \mathcal{B}(W) \to \mathbf{U}^-$ in Corollary 3.9 (3) induces an isomorphism $\big(\mathcal{B}(W)/\big(\sum_{i=1}^{\theta} \mathcal{B}(W) w_i^{m_i+1}\big)\big)_\alpha \cong \big(\mathbf{U}^-/\big(\sum_{i=1}^{\theta} \mathbf{U}^- F_i^{m_i+1}\big)\big)_{-\alpha}$ for all $\alpha \in \mathbb{N}^{\mathbb{I}}$ if we define $\deg(w_i) = \alpha_i$ for all i.

By assumption on the braiding (and since $\mathbb{I}$ is connected), $q_{ii} = q^{2d_i}$ for all $1 \leq i \leq \theta$, where $0 \neq q \in \Bbbk$ is not a root of unity. The braiding matrix $(q_{ji}^{-1})_{1 \leq i,j \leq \theta}$ of W with respect to the basis $w_1, \dots, w_\theta$ is twist equivalent to $(q^{-d_i a_{ij}})$. Let $\widehat{W}$ be the braided vector space with braiding matrix $(q^{-d_i a_{ij}})$ with respect to a basis $\widehat{w_1}, \dots, \widehat{w_\theta}$. By [AS1, Proposition 3.9, Remarks 3.10] $\mathcal{B}(W)/\big(\sum_{i=1}^{\theta} \mathcal{B}(W) w_i^{m_i+1}\big) \cong \mathcal{B}(\widehat{W})/\big(\sum_{i=1}^{\theta} \mathcal{B}(\widehat{W}) \widehat{w_i}^{m_i+1}\big)$ as $\mathbb{Z}^{\mathbb{I}}$-graded vector spaces, where $\deg(\widehat{w_i}) = \alpha_i$ for all i. The claim now follows from Remark 4.14.

Now let J be a connected component of $\mathbb{I}$ and $\mathbb{I}' = \mathbb{I} \setminus J$. Let $\mathbf{U}_J$ be the subalgebra of $\mathbf{U}$ generated by Γ and $E_j, F_j, j \in J$. Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ generated by Γ and $E_i, F_i, i \in \mathbb{I}'$. We assume by induction that $L_{\mathbf{U}'}(\chi)$ satisfies (1), (2) and (3).

We first show that $L(\chi)$ satisfies (2). Let $m \in L(\chi)$ be a weight vector of weight $\chi' \in \widehat{\Gamma}$ such that $E_i m = 0$ for all $i \in \mathbb{I}$. By Lemma 4.13 (1) Φ induces a linear isomorphism of weight spaces

$$\bigoplus_{\substack{\chi^{-1}\varphi\psi = \chi' \\ \varphi,\psi \in \widehat{\Gamma}}} L_{\mathbf{U}_J}(\chi)^{\varphi} \otimes L_{\mathbf{U}'}(\chi)^{\psi} \to L(\chi)^{\chi'}.$$

Hence there are finitely many elements $m_l \in L_{\mathbf{U}_J}(\chi)^{\varphi_l}$, and $m_l' \in L_{\mathbf{U}'}(\chi)^{\psi_l}$, $1 \leq l \leq n$, with $\varphi_l, \psi_l \in \widehat{\Gamma}, \varphi_l \psi_l = \chi\psi$ for all $1 \leq l \leq n$ such that $m_1', \dots, m_n'$ are $\Bbbk$-linearly independent and $\Phi(\sum_{l=1}^n m_l \otimes m_l') = m$. By Lemma 4.13 (2)

$$\Phi\Big(\sum_{l=1}^{n} (\chi_{-1}\psi_l)(K_j) E_j m_l \otimes m_l'\Big) = \sum_{l=1}^{n} E_j \Phi(m_l \otimes m_l') = E_j m = 0.$$

for all $j \in J$. Since Φ is bijective, and the elements m_l' are linearly independent it follows that $E_j m_l = 0$ for all $j \in J$ and $1 \leq l \leq n$. By (2) for $\mathbf{U}_J$, and since $m_l \in L_{\mathbf{U}_J}(\chi)^{\varphi_l}$ for all l, the elements m_l are scalar multiples of $\overline{1} \in L_{\mathbf{U}_J}(\chi)$. Therefore $m = \Phi(\overline{1} \otimes m'')$, where $m'' \in L_{\mathbf{U}'}(\chi)^{\chi'}$. Then by Lemma 4.13 (3), $0 = E_i m = \Phi(\overline{1} \otimes E_i m'')$, hence $E_i m'' = 0$ for all $i \in \mathbb{I}'$; thus, m'' is a scalar multiple of $\overline{1}$ by (2) for $\mathbf{U}'$. Hence $m \in \Bbbk\overline{1}$ and $\chi' = \chi$.

We next show that (2) for $L(\chi)$ implies (1). Let $0 \neq M \subset L(\chi)$ be a $\mathbf{U}$-submodule, and let $0 \neq m \in M$. Then $\mathbf{U}_J m$ is a finitely generated

$\mathbf{U}_J$-submodule of $L(\chi)$. By Lemmas 4.3 and 4.5 (3), there is $u \in U_J$ such that um is an element of maximal weight in $\mathbf{U}_J m$ and $E_j um = 0$ for all $j \in J$. Then $\mathbf{U}'um$ is $\mathbf{U}'$-finitely generated and by the same reason there is an element $u' \in \mathbf{U}'$ such that $u'um$ is an element of maximal weight in $\mathbf{U}'um$ and $E_i u'um = 0$ for all $i \in \mathbb{I}'$. Then $u'um$ is a $\Bbbk$-linear combination of elements of the form $F_{i_1} \cdots F_{i_n} E_{l_1} \cdots E_{l_m} gum$, where $n, m \geq 0$, $i_1, \dots, i_n$, $l_1, \dots, l_m \in \mathbb{I}'$ and $g \in \Gamma$. Since the elements $F_{i_1}, \dots, F_{i_n}$ and $E_{l_1}, \dots, E_{l_m}$ commute or skew-commute with E_j for all $j \in J$, it follows that $E_j u'um = 0$ for all $j \in J$. Let χ' be the weight of $u'um$. Since (2) holds for $L(\chi)$ it follows that $\chi' = \chi$, and $u'um \in \Bbbk\ell_\chi$. Hence $\ell_\chi \in M$, and $M = L(\chi)$ since ℓ_χ generates the $\mathbf{U}$-module $L(\chi)$.

Finally (3) for $L(\chi)$ follows from the isomorphism Φ in Lemma 4.13. $\square$

For an algebra A we denote the set of isomorphism classes of finite-dimensional left A-modules by $\operatorname{Irr}(A)$.

Corollary 4.16. (1) *The map*

$$\widehat{\Gamma}^+ \to \{[L] \mid L \in \mathcal{C}^{hi}, L \textit{ integrable and simple}\},$$

defined by $\chi \mapsto [L(\chi)]$, *is bijective.*

(2) *Assume that the Cartan matrix of* $\mathcal{D}_{red}$ *is of finite type. Then the map in* (1) *defines a bijection* $\widehat{\Gamma}^+ \to \operatorname{Irr}(\mathbf{U})$.

Proof. (1) The map is well-defined and injective by Corollary 4.9 and Theorem 4.15. To prove surjectivity let $L \in \mathcal{C}^{hi}$ be integrable and simple. By Lemma 4.5 L has a maximal weight χ, and by Lemma 4.3 and Proposition 4.8 $L \cong L(\chi)$.

(2) By Lemma 4.13 and the arguments in the proof of Theorem 4.15 it suffices to assume that the braiding matrix is of the form $(q^{d_i a_{ij}})$, where $(d_i a_{ij})$ is the symmetrized Cartan matrix of finite type and $0 \neq q \in \Bbbk$ is not a root of unity. Then the claim follows from [J, 5.9, 5.15. 6.26]. $\square$

4.5. **Complete Reducibility Theorems.**

Here is one of the main results of the present paper extending [L2, 6.2.2], the analogue of (b) in the Introduction.

Theorem 4.17. *Let* M *be an integrable module in* $\mathcal{C}^{hi}$. *Then* M *is completely reducible and* M *is a direct sum of* $L(\chi)'s$ *where* $\chi \in \widehat{\Gamma}^+$.

Proof. By Theorem 4.15 it suffices to show that M is completely reducible. We proceed by induction on the number of connected components of $\mathbb{I}$.

Let $\mathbb{I}$ be connected. We may assume that $M \neq 0$. We need only show that M is a sum of simple $\mathbf{U}$-submodules. Thus we may further assume that M is $\mathbf{U}$-finitely generated, and $M = M_C$ for some coset C of Q. By Proposition 4.12 (1) the operator Ω_G for C is locally finite. Since generalized eigenspaces of module endomorphisms are submodules we may assume that M is a generalized eigenspace of Ω_G with eigenvalue λ.

Let N be a proper $\mathbf{U}$-submodule of M. It suffices to show that there exists a simple $\mathbf{U}$-submodule S such that $S \cap N = 0$. Then M has a simple submodule (take $N = 0$), and M must be the sum of all simple submodules (take N to be this sum).

Let $m \in M \setminus N$ and set $L = \mathbf{U} \cdot m$. Then $L/(N \cap L) \neq 0$ is finitely generated and has a maximal weight χ by Lemma 4.5 (3). Since χ is also a weight for L there is a maximal weight χ' for L which satisfies $\chi \leq \chi'$. By the characterization of the eigenvalues of Ω_G in Proposition 4.12 (3) we have that $G(\chi) = \lambda = G(\chi')$. By Proposition 4.8 and Lemma 4.3 both characters χ and χ' are dominant. Hence $\chi = \chi'$ by Lemma 4.10 (2). Therefore χ is a maximal weight for L. The projection $L \to L/(N \cap L)$ induces a surjection $L^\chi \to (L/(L \cap N))^\chi$. We choose $\ell \in L^\chi \setminus N$. Then $S = \mathbf{U} \cdot \ell$ is simple by Lemma 4.3, Proposition 4.8 and Theorem 4.15, and $S \cap N = 0$.

In the general case let J be a connected component of $\mathbb{I}$ and $\mathbb{I}' = \mathbb{I} \setminus J$. Let $\mathbf{U}_J$ be the subalgebra of $\mathbf{U}$ generated by Γ and E_j, F_j, $j \in J$. Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ generated by Γ and E_i, F_i, $i \in \mathbb{I}'$. We assume that any integrable $\mathbf{U}'$-module in the category $\mathcal{C}^{hi}$ for $\mathbf{U}'$ is completely reducible.

Let M be a finitely generated and integrable $\mathbf{U}$-module in $\mathcal{C}^{hi}$, and let $N \subset M$ be a proper $\mathbf{U}$-submodule. As before it suffices to show that there exists a simple $\mathbf{U}$-module $S \subset M$ such that $N \cap S = 0$. By the first part of the proof M is completely reducible over $\mathbf{U}_J$. Hence there exists a simple U_J-submodule $S_1 \subset M$ such that $N \cap S_1 = 0$. Let $m \in S_1$ with $S_1 = \mathbf{U}_J m$. Since $\mathbf{U}'m \not\subset N$ and $\mathbf{U}'m$ is completely reducible by induction there is a simple $\mathbf{U}'$-module $S_2 \subset \mathbf{U}'m$ such that $S_2 \cap N = 0$. By Theorem 4.15 there is a character $\chi \in \widehat{\Gamma}$ and an element $u \in \mathbf{U}'^\chi$ such that $S_2 = \mathbf{U}'um$ and $E_i um = 0$ for all $i \in \mathbb{I}'$. As in the proof of Theorem 4.15 it follows that $E_j um = 0$ for all $j \in J$. By Proposition 4.8 and Theorem 4.15 $S := \mathbf{U}um$ is simple over $\mathbf{U}$. Moreover $S \cap N = 0$, since S is $\mathbf{U}$-simple and $S \not\subset N$. $\square$

4.6. **Reductive pointed Hopf algebras.**

Let A be an algebra and $B \subset A$ a subalgebra. We say that

> A is *reductive* if any finite-dimensional left A-module is completely reducible.
>
> A is B-*reductive* if every finite-dimensional left A-module which is B-semisimple when restricted to B is A-semisimple.

A pointed Hopf algebra H with $\Gamma = G(H)$ is called Γ-reductive if it is $\Bbbk\Gamma$-reductive. Compare with [KSW1, KSW2].

Corollary 4.18. $\mathbf{U}$ *is* Γ*-reductive.*

Proof. This is a special case of Theorem 4.17 since finite-dimensional $\mathbf{U}$-modules which are completely reducible over $k\Gamma$ are integrable objects of $\mathcal{C}^{hi}$. $\square$

In Theorem 4.21, we shall need a generalization of Lemma 4.7. Suppose $q \in \Bbbk^\times$ is not a root of unity. As usual, we define

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad \begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a][a-1]\cdots[a-n+1]}{[1][2]\cdots[n]}, \quad [n]^! = [1][2]\cdots[n]$$

for all $a, n \in \mathbb{Z}$ and $n > 0$, and $\begin{bmatrix} a \\ 0 \end{bmatrix} = 1,\ [0]^! = 1$.

Lemma 4.19. *Let $0 \neq \ell \in \Bbbk$. Let A be an algebra with elements E, F, K, L such that K and L are invertible and*

$$KL = LK, \tag{4.13}$$

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2}F, \tag{4.14}$$

$$LEL^{-1} = q^2 E, \qquad LFL^{-1} = q^{-2}F, \tag{4.15}$$

$$EF - FE = \ell(K - L^{-1}). \tag{4.16}$$

Let $(K, L; a) = \dfrac{Kq^a - L^{-1}q^{-a}}{q - q^{-1}}$, $a \in \mathbb{Z}$.

(1) *For all $r, s \in \mathbb{Z}, r, s \geq 0$,*

$$E^r F^s = \sum_{i=0}^{\min(r,s)} F^{s-i} h_i(r,s) E^{r-i}, \textit{ where}$$

$$h_i(r,s) = \ell(q - q^{-1}) \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ i \end{bmatrix} [i]^! \prod_{j=1}^{i} (K, L; i - (r+s) + j).$$

(2) *KL acts semisimply on any finite-dimensional left A-module.*

Proof. By rescaling E we may assume that $\ell = (q - q^{-1})^{-1}$. Let $\lambda \in \Bbbk$ be an eigenvalue of K on M. Then by (4.15) for any natural number $r > 0$, F^r maps the generalized eigenspace of λ with respect to K into $\cup_{n \geq 1} \ker(K - \lambda q^{-2r})^n$. Since q is not a root of unity there is an integer $s > 0$ such that $F^s M = 0$. The elements $(K, L; a)$ satisfy the same rules as $[K; a] = [K, K; a]$ in [J]. One can check that the proofs of (1) in [J, Lemma 1.7] and of (2) in [J, Prop. 2.3]– which uses (1)– work in our more general situation. In particular, $\left(\prod_{j=-(s-1)}^{s-1} (KL - q^{-2j})\right) M = 0$. □

Remark 4.20. The following is a standard fact in abelian group theory. Let A be a subgroup of an abelian group B with $[B : A] < \infty$. If M is a $\Bbbk B$-module such that $M_{|\Bbbk A}$ is semisimple then M is semisimple.

Proof. If $\lambda \in \widehat{A}$, then we denote by M_λ the isotypic component of type λ. Then $M = \oplus_{\lambda \in \widehat{A}} M_\lambda$ and each M_λ is a $\Bbbk B$-submodule. Thus, we can assume that $M = M_\lambda$ for some λ. There exists $\Lambda \in \widehat{B}$ extending λ since the multiplicative group of the algebraically closed field $\Bbbk$ is a divisible hence injective abelian group. Then A acts trivially on $M' = M \otimes \Bbbk_{\Lambda^{-1}}$, and this

becomes a module over the finite group B/A. Hence M' is a semisimple $\Bbbk B$-module, and so is $M \simeq M' \otimes \Bbbk_\Lambda$. □

Recall that Γ^2 denotes the subgroup of Γ generated by the products $K_1L_1, \dots, K_\theta L_\theta$, see Definition 3.19.

Theorem 4.21. *The following are equivalent:*

(1) $\mathbf{U}$ *is reductive.*
(2) $[\Gamma : \Gamma^2]$ *is finite.*

If $\mathbf{U}$ *is reductive, then the Cartan matrix of* $\mathcal{D}_{red}$ *is invertible.*

Proof. Suppose $\mathbf{U}$ is reductive. There is a well-defined surjective algebra map $\mathbf{U} \to k[\Gamma/\Gamma^2]$ mapping all E_i and F_i onto zero and any $g \in \Gamma$ onto its residue class in Γ/Γ^2. Hence the group algebra $k[\Gamma/\Gamma^2]$ is reductive, and Γ/Γ^2 must be finite.

Conversely suppose that Γ/Γ^2 is finite. Let M be a finite-dimensional left $\mathbf{U}$-module. Then for any $1 \leq i \leq \theta$, the elements E_i, F_i, K_i, L_i in $\mathbf{U}$ satisfy the assumptions of Lemma 4.19. Hence K_iL_i acts semisimply on M by Lemma 4.19. Then we obtain from Remark 4.20 that Γ acts semisimply on M. Thus M is a semisimple $\mathbf{U}$-module by Corollary 4.18.

Finally, if (1) and (2) hold, then (a_{ij}) is invertible by Lemma 3.20 (3). □

5. A characterization of quantum groups

We now turn to the representation theory of the more general pointed Hopf algebra $\mathcal{U}(\mathcal{D}, \lambda)$ where $\mathcal{D}$ is generic and of finite Cartan type. Let $\mathcal{D}'$ be the datum with perfect linking parameter λ' associated to the set $\mathbb{I}^s$ of all non-linked vertices as in Theorem 2.1. Then $\mathcal{U}(\mathcal{D}', \lambda') \cong \mathcal{U}(\mathcal{D}_{red}, \ell) := \mathbf{U}$, where $(\mathcal{D}_{red}, \ell)$ is the reduced datum and its linking parameter associated to $(\mathcal{D}', \lambda')$ as in Lemma 3.5. Thus $\mathcal{D}_{red}$ is generic of finite Cartan type, hence regular by Lemma 3.20. By Section 2 there is a projection of Hopf algebras

$$\pi_{\mathcal{D}} : \mathcal{U}(\mathcal{D}, \lambda) \to \mathbf{U}.$$

We may consider then any $\mathcal{U}(\mathcal{D}_{red}, \ell)$-module as a $\mathbf{U}$-module via π_D, and $\pi_{\mathcal{D}}$ induces a mapping $\pi_{\mathcal{D}}^*$ from the isomorphism classes of $\mathbf{U}$-modules to the isomorphism classes of $\mathcal{U}(\mathcal{D}, \lambda)$-modules. Let $\Gamma^2 \subset \Gamma$ be the subgroup defined in Definition 3.19 for $\mathcal{D}_{red}$.

Proposition 5.1. *Let* $\mathcal{D}$ *be a generic datum of finite Cartan type with linking parameter* λ *and define* $\pi_{\mathcal{D}}$ *as above. Then*

$$\widehat{\Gamma}^+ \to \operatorname{Irr}((\mathcal{U}(\mathcal{D}, \lambda)), \ \chi \mapsto \pi_{\mathcal{D}}^*[L(\chi)],$$

is bijective.

Proof. This follows from Corollary 4.16 and [RS2, Theorem 4.6]. □

Lemma 5.2. *Let $\mathcal{D}$ be a generic YD-datum of finite Cartan type and abelian group Γ, and let λ be a linking parameter for $\mathcal{D}$. Let $h \in \mathbb{I}$, and assume that h is not linked. Define $\mathcal{D}', \lambda'$, $\mathcal{U} = \mathcal{U}(\mathcal{D}, \lambda)$, $\mathcal{U}' = \mathcal{U}(\mathcal{D}', \lambda')$ and $K = \mathcal{U}^{co\mathcal{U}'}$ as in Theorem 2.1 for $L = \{h\}$. Then $M = \mathcal{U}/(\mathcal{U}\mathcal{U}'^{+} + \mathcal{U}(K^{+})^2)$ is a finite-dimensional vector space.*

Proof. Let J be the connected component of $\mathbb{I}$ containing h, and let $X_J = \oplus_{i \in J} \Bbbk x_i$. Then the natural algebra map $\rho : \mathcal{B}(X) \to \mathcal{U}$ is injective by Theorem 1.11. We view ρ as an inclusion. By Theorem 2.1 K is contained in $\mathcal{B}(X_J)$. Since $K = \mathcal{U}^{co\mathcal{U}'}$ is a left coideal subalgebra of $\mathcal{U}$, it follows that K is a left coideal subalgebra of $\mathcal{B}(X_J)\Bbbk\Gamma \cong \mathcal{B}(X_J)\#\Bbbk\Gamma$. Hence $K \subset \mathcal{B}(X_J)$ is a left coideal subalgebra in the braided sense, that is, $\Delta_{\mathcal{B}(X_J)}(K) \subset \mathcal{B}(X_J) \otimes K$. Moreover, K is $\mathbb{N}^J$-graded by Theorem 2.1. Since the braiding of X_J is of finite Cartan type, it follows from [HS, Corollary 6.16] that there are finitely many $\mathbb{N}^J$-homogeneous elements $a_1, \dots a_m \in K$ such that for all $i \in J$ the subalgebra $\Bbbk(a_i)$ is isomorphic to $\mathcal{B}(\Bbbk a_i)$, and the multiplication map $\Bbbk(a_m) \otimes \cdots \Bbbk(a_1) \to K$ is bijective. Since $\mathcal{B}(X_J)$ is an integral domain (see for example [AS3, Theorem 4.3]), for each i the Nichols algebra $\mathcal{B}(\Bbbk a_i)$ is a polynomial ring. Hence the elements $a_m^{n_m} \cdots a_1^{n_1}, n_1, \dots, n_m \geq 0$, form a $\Bbbk$-basis of K. The existence of such a PBW-basis can also be derived from [Kh]. Since M is an epimorphic image of $K/(K^{+})^2$ by the decomposition $K\#\mathcal{U}' \cong \mathcal{U}$ in Theorem 2.1, it follows that M is the $\Bbbk$-span of the images of $a_1, \dots, a_m$ thus finite-dimensional. □

Theorem 5.3. *Let $\mathcal{D}$ be a generic YD-datum of finite Cartan type with abelian group Γ, and let λ be a linking YD-datum for $\mathcal{D}$.*

(i) *The following are equivalent:*

(1) $\mathcal{U}(\mathcal{D}, \lambda)$ *is Γ-reductive.*
(2) *The linking parameter λ of $\mathcal{D}$ is perfect.*

(ii) *The following are equivalent:*

(1) $\mathcal{U}(\mathcal{D}, \lambda)$ *is reductive.*
(2) (a) *The linking parameter λ of $\mathcal{D}$ is perfect.*
(b) $[\Gamma : \Gamma^2]$ *is finite.*

Proof. (i) We assume that $U(\mathcal{D}, \lambda)$ is Γ-reductive, and that the linking is not perfect. We choose an element $h \in \mathbb{I}$ which is not linked and define $L = \{h\}$. Let $M = \mathcal{U}/(\mathcal{U}\mathcal{U}'^{+} + \mathcal{U}(K^{+})^2)$ as in Lemma 5.2. Then M is finite-dimensional by Lemma 5.2. By Corollary 2.2 $x_h M \neq 0$. Hence M is not semisimple since by [RS2, Theorem 4.6] any finite-dimensional simple $U(\mathcal{D}, \lambda)$-module is annihilated by x_h.

To obtain a contradiction we finally show that M is semisimple as a Γ-module by restriction. The vector space $\mathcal{U}(\mathcal{D}, \lambda)$ is the $\Bbbk$-span of elements of the form xh, x a monomial in the elements $x_1, \dots, x_\theta$, $h \in \Gamma$. Let $g \in \Gamma$, then $gxh = \chi(g)xhg$ for some $\chi \in \widehat{\Gamma}$. Hence in the module M, we have

$g\overline{xh} = \chi(g)\overline{xhg} = \chi(g)\overline{xh}$ since $g-1 \in U'^{+}$. Thus M as a Γ-module is the sum of weight spaces.

Conversely assume that the linking parameter is perfect. Then $U(\mathcal{D},\lambda) \cong U(\mathcal{D}_{red},\ell)$ for some generic reduced YD-datum. Since the Cartan matrices of $\mathcal{D}$ and $\mathcal{D}_{red}$ are of finite type, $\mathcal{D}_{red}$ is regular by Lemma 3.20 (2), and $U(\mathcal{D}_{red},\ell)$ is Γ-reductive by Corollary 4.18.

(ii) follows from the argument in the proof of (i) and Theorem 4.21. $\square$

Theorem 5.3 combined with [AA, Theorem 1.1], that generalizes the main result of [AS3], gives immediately the following characterization of quantized enveloping algebras.

Theorem 5.4. *Let H be a pointed Hopf algebra with finitely generated abelian group $G(H)$, and generic infinitesimal braiding. Then the following are equivalent:*

1. *H is a Γ-reductive domain with finite Gelfand-Kirillov dimension.*
2. *The group $\Gamma := G(H)$ is free abelian of finite rank, and there exists a reduced generic datum of finite Cartan type $\mathcal{D}_{red}$ for Γ with linking parameter ℓ such that $H \simeq \mathbf{U} = \mathcal{U}(\mathcal{D}_{red},\lambda)$ as Hopf algebras.*

If H satisfies (2), then H is reductive iff $[\Gamma : \Gamma^2]$ is finite.

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